Society for Economics Research in India
Working Paper Series

TOO COSTLY TO FOLLOW BLINDLY: ENDOGENOUS LEARNING AND HERDING

Srijita Ghosh
(Ashoka University, Sonepat)

Working Paper No. 4
http://seri-india.org/research

September 2020

Too Costly To Follow Blindly: Endogenous Learning and Herding *

Srijita Ghosh†

September 2020

Abstract

I analyze the impact of endogenizing social and private learning in a *herding* problem. Private learning is modeled à la *rational inattention* literature. I find a non-monotone relationship between social and private learning. They are substitutes when private learning is sufficiently cheap and become complement for higher private learning costs and eventually becomes uninformative. This happens because an increase in private learning costs makes social learning less informative. As an implication, only the reduction of the cost of private learning unambiguously increases welfare contrary to the *herding* result, where restricting social learning initially is optimal.

Keywords: Herding, Rational Inattention, Social Learning
JEL Code: D82, D83, D89

*I greatly benefited from the comments and suggestions of Andrew Caplin, Debraj Ray, David Pearce, Parikshit Ghosh. I am also grateful to seminar participants at EEA 2019, ACEGD ISI Delhi 2016, SERI 2020 for their comments and suggestions. All mistakes are my own.

†Economics Department, Ashoka University, email: srijita.ghosh@ashoka.edu.in

1 Introduction

In several economic decision making problems agents deliberate before choosing an alternative. Starting with Banerjee (1992) and Bikhchandani et al. (1992) several papers have analyzed the impact of social learning on decision making when agents also get private signals exogenously. This literature had examined a feature of the dynamics of learning where if the first few generations of agents choose the same action then subsequent generations will ignore their private signals and mimic the action of their predecessors. This phenomenon, known as *herding*, generates a policy prescription of suppressing social learning in earlier generations.

A natural question to ask here is what if agents do not exogenously learn about actions from all their predecessors or get a private signal but choose how much social or private information they want to gather, how would that change the equilibrium? How does individual information choice get affected by the amount of information available in society? Is herding robust to such endogenous learning? What would be a welfare-improving policy?

For example, consider an undergraduate student deciding her major choice. The major choice had a significant effect on lifetime earning, but at the time of the choice the student may not know what major would suit her best and improve her expected lifetime earnings, and once the decision is made it is difficult to change. In that case, the student can ask her friends about their major choice and do some exploration on her own, before making a choice. The choice problem outlined here predicts the optimal learning decision for major choice given the student faces a cost of learning.

To evaluate the impact of endogenizing learning, I use the following ingredients: a mechanism for private learning, a mechanism for social learning, and a model of social connectivity. Private learning is informative about idiosyncratic types and is subject to a cost à la *rational inattention* literature. The cost ensures agents do not necessarily learn fully about their types.

The social learning model is similar to herding literature. In every generation, a large but finite number of agents enter. Generation t agents choose action in period t and the $t+1$ generation can observe the actions chosen by their immediate predecessors in generation t. Similar to the herding literature future generation agents can only observe the *action* of their predecessors and not the belief or realized payoff. Social learning is thus informative about the distribution of types in the economy.

A general theory of social connection structure would require specifying the social network. For simplicity, we abstract away from network structures and assume that observing more agents is weakly more costly. For a more connected society, the marginal cost of social learning would be lower.

In this framework, I solve for the optimal learning strategy for a one-time discrete choice problem. Before solving for the optimal learning strategy I show the optimal learning control would be to choose social learning first, followed by private learning. Thus the information that the DM obtains from social learning can affect her private learning strategy. This implies it is possible to obtain a herd if social learning effectively stops the DM from learning privately.

However, in herding models, once sufficiently many agents choose one action, others ignore their private signals and mimic, making the two types of learning substitutable. But in herding models, there are no costs of any form learning it is not possible to analyze whether the DM would decrease one form of learning when the cost of other forms of learning decreases, i.e., substitutes in terms of standard price theory. Here, I analyze whether agents increase or decrease their optimum level of social learning as the marginal cost of private learning increases, i.e., would social and private learning remain *substitutes* or become *complements*?

I show, for a given weakly increasing and weakly convex cost of social learning function, social learning change *non-monotonically* with the marginal cost of private learning. Fixing a social learning cost function, there exists a threshold below which the two types of learning are substitutes and above which they become complements. When the cost of private learning is very high, agents stop all forms of learning. The main intuition is, as the marginal cost of private learning increases, Bayesian agents correctly update that social learning is less valuable since other agents also learn less privately. When this effect starts to dominate the relative cost advantage, the two types of learning become complements.

Unlike the herding literature, restricting social learning initially is not necessarily welfare improving. When the two types of learning are complements reducing access to social learning makes agents weakly worse off. However, reducing the cost of private learning unambiguously increases total welfare in the economy.

1.1 Literature Review

The *rational inattention* literature considers the discrete choice problem of a decision-maker subject to costly information. Following Sims (2003), Matějka and McKay (2015) modeled the cost of learning as a linear function of the *Shannon's relative entropy* between the prior and the posterior belief and showed the optimal stochastic choice takes the form of multinomial logit. Caplin and Dean (2015) gives axiomatic characterization for costly information acquisition problems. Caplin et al. (2019) showed that rationally inattentive behavior implies consideration sets. Caplin et al. (2015) combined exogenous social learning of market share to a model of rational inattention. They found that observing the market share affects the private learning, and subsequently the optimal behavior of the agent in the model.

Banerjee (1992) showed that herding is an equilibrium, albeit, inefficient and Bikhchandani et al. (1992) showed a small change in the initial condition can lead to different information cascade. Park and Sabourian (2011) discusses a *rational herding* model where agents herd when information is sufficiently dispersed.

There is another literature that this paper connects to philosophically, though the models considered are significantly different. Several authors, in various contexts, ranging from finance to labor market decision, have shown how either existing knowledge or social learning affects the learning and decision making behavior of the DM. For example, in the context female labor force participation, Fernández (2013) showed that how social learning in form of culture affects the labor market entry decision for women, Fogli and Veldkamp (2011) showed that women observe the choice made by others in the neighborhood along with her mother's labor market decision, which affects her own labor market decision. In a different example, Van Nieuwerburgh and Veldkamp (2009) showed how the learning behavior changes depending on whether the origin of an asset, home or foreign. Similarly in this model, DM optimally chooses to learn socially first, which distorts his choice of the private learning and subsequently his decision.

The rest of the paper is arranged as follows. Section 2 describes the two cost structures and sets up the baseline model. In section 3, I solve the agent's optimization problem to show the non-monotonicity result and discuss the herding behavior, and section 4 concludes. All extensions and additional proofs are in the appendix.

2 Model

2.1 Environment

Consider an infinite horizon economy in discrete time, i.e. $t \in \{0, 1, \ldots, \infty\}$. At each period t a large but finite number of agents, N, enter the economy. At any period $t \geq 0$ when an agent enters the economy he chooses to learn, takes an irreversible action, and leaves the economy never to come back again.

Let $A = \{a, b\}$ be the set of actions. Agent doesn't know his payoff from choosing action $i \in A$. Let $\Omega = \{\omega_1, \omega_2\}$ be the set of all possible strict rankings of payoffs in A, where $\omega_1 \equiv a \succ b$ and $\omega_2 \equiv b \succ a$. Let $\Gamma \equiv \Delta(\Omega)$ be the set of possible distributions over Ω.

Let $\Delta(\Gamma)$ denote the set of all possible distributions over Γ. At any period $t \geq 0$ agents enter with a common prior $\gamma_0 \in \Delta(\Gamma)$. After entering, the agent tries to learn about his own type ω_i and chooses takes an alternative $i \in A$. Let μ^* denote the true distribution of types where $\mu^* \in int(\gamma_0)$.

Let $u : A \times \Omega \to \mathbb{R}$ be the state dependent utility function. Consider,

$$\begin{aligned} u(a, \omega_1) = u(b, \omega_2) = \bar{u} \\ u(a, \omega_2) = u(b, \omega_1) = \underline{u} \end{aligned} \tag{1}$$

where $\bar{u} > \underline{u}$, so type ω_1 gets a higher payoff from action a and type ω_2 gets a higher payoff from action b. Define $\Delta u = \bar{u} - \underline{u}$, the gain in payoff by matching over mismatching the state. Assume that agents are Bayesian expected utility maximizers.

2.2 Costly learning

Agents can learn privately and socially. The social learning is informative about the distribution of types $\mu \in \Gamma$, and private learning is informative about own type, $\omega \in \Omega$.

2.2.1 Private learning

Let $\pi(s,\omega)$ be an information structure that generates a distribution of posterior beliefs $\gamma_\pi \in \Delta(\Gamma)$. Then for any prior belief $\mu \in \Gamma$ the posterior belief for any state ω given signal s would be

$$Pr(\omega|s) = \frac{\pi(s|\omega)\mu(\omega)}{\sum_{\omega'} \pi(s|\omega')\mu(\omega')}$$

Let us assume for any $\mu \in supp(\gamma)$, $\pi(s,\omega|\mu) = \pi(s,\omega)$, i.e., the signal structure is independent of the prior belief μ. If two distinct signals generate the same posterior belief then they are equally Blackwell informative (Blackwell et al. (1953)). Since more signals are weakly more costly it is optimal to choose an unique signal to generate a posterior belief. Hence, the probability of any posterior belief is

$$\gamma_\pi(Pr(\omega|s)) = \sum_{\omega} \pi(s|\omega)\mu(\omega) = P_\pi(s).$$

By Bayes Plausibility (refer Kamenica and Gentzkow (2011), Matějka and McKay (2015))

$$\sum_{s} P(\omega|s)\gamma_\pi(P(\omega|s)) = \mu(\omega).$$

Choosing any γ is hence equivalent to choosing an information structure

$$\pi(s|\omega) = \frac{P(\omega|s)\gamma(P(\omega|s))}{\mu(\omega)}.$$

By similar logic of Blackwell informativeness two distinct posteriors cannot generate same distribution over actions. Hence choosing a distribution of posterior probability of actions is equivalent to choosing an information structure.

The cost of private learning is given by Shannon's relative entropy between the prior and the posterior probability of choice (Cover and Thomas (2012)). Let $P\left(i,\omega|\mu\right)$ be the posterior probability of choosing action $i \in A$ when type is $\omega \in \Omega$ and prior $\mu \in \Gamma$. Define $P\left(i|\mu\right) \equiv \sum_{\omega\in\Omega} \mu\left(\omega\right) P\left(i,\omega|\mu\right)$ as the prior probability of choosing action $a \in A$. The cost function is given by,

$$C\left(\lambda,\mu\right) = \lambda \left\{ \underbrace{\sum_{\omega\in\Omega} \mu\left(\omega\right) \sum_{a\in A} P\left(a,\omega|\mu\right) \ln P\left(a,\omega|\mu\right)}_{\text{expected entropy of the posterior distributions}} - \underbrace{\sum_{a\in A} P\left(a|\mu\right) \ln P\left(a|\mu\right)}_{\text{entropy of the prior distribution}} \right\} \tag{2}$$

where $\lambda \in [0,\infty]$ be the marginal cost of private learning [1].

The specific form of the private cost function simplifies our analysis, however, there are some

[1] We can also write $C\left(\lambda,\mu\right) = \lambda D(P(a|\mu)||P(a,\omega|\mu))$

interesting properties of the cost function that makes it an appropriate choice here. First, the cost function belongs to a class of cost function, named, Posterior Separable (PS) (refer Caplin, Dean, and Leahy (2018)) which allows the cost to be dependent on the posterior only. Second, the cost function allows the cost to be increasing in *precision* without any distributional assumption on the prior or signal structure. Third, the cost of a learning strategy depends on the prior. This captures the notion that with a sufficiently confident prior the cost of learning becomes relatively more expensive for further learning. This will make it easier for a herding equilibrium to exist.

2.2.2 Social learning

Any agent at any period $t \geq 1$ can observe the action of any $t-1$ generation agents subject to a cost. The cost of social learning $c(n)$, where n be the number of $t-1$ generation agents that an agent in generation t observes, has the following properties,

$$\begin{aligned} & c(n) \geq 0, \quad 0 \leqslant n \leqslant N, \quad c(N) > \bar{u} \\ & c(n) \leq c(n+1), \quad c(n) - c(n+1) \leq c(n+1) - c(n), \quad 0 \leqslant n \leqslant N-1 \end{aligned} \tag{3}$$

i.e., $c(n)$ in non-negative, weakly monotone and weakly convex, and observing everyone are never optimal. Once the t generation agent decides n, he randomly picks n agents from generation $t-1$ and observes there action is a block. In the online appendix, I show the consequence of sequential learning.

Let x_n denote the distribution of action a chosen by n agents. The Bayesian agent, given a belief γ updates her belief to $\gamma_{x_n} \in \Delta(\Gamma)$ upon observing x_n. Here, he accounts for possible mismatch between state and action chosen by his predecessors.

2.3 Time 0 agents

The $t=0$ agent has only the option of learning privately. Let $\mu_0(\omega) = E_{\gamma_0}(\mu(\omega))$ the optimization problem of a $t=0$ agent is given by,

$$V(A, \mu_0) = \max_{P(i,\omega|\mu_0)} \sum_{\omega\in\Omega} \mu_0(\omega) P(i,\omega|\mu_0) u(i,\omega) - C(\lambda, \mu_0). \tag{4}$$

Following **??**, the solution to the agent's optimization problem would be

$$P(i,\omega|\mu_0) = \frac{P(i|\mu_0) e^{\frac{u(i,\omega)}{\lambda}}}{\sum_{j\in A} P(j|\mu_0) e^{\frac{u(j,\omega)}{\lambda}}} \quad \forall i \in A, \omega \in \Omega \tag{5}$$

The Bayesian plausibility implies given their prior γ_0,

$$\sum_{\omega\in\Omega} \mu_0(\omega_i) \frac{\exp(u(i,\omega)/\lambda)}{\sum_{j\in A} P(j|\gamma) \exp(u(j,\omega)/\lambda)} \leq 1 \quad \forall \quad i \in A. \tag{6}$$

The inequality holds with equality if $P(i|\gamma) > 0$.

Using equation 6 for both $a, b \in A$ we get,

$$P(a|\mu_0) = \begin{cases} \frac{\mu_0(\omega_1)\exp(\bar{u}/\lambda) - \mu_0(\omega_2)\exp(\underline{u}/\lambda)}{\exp(\bar{u}/\lambda) - \exp(\underline{u}/\lambda)} & \text{if} \quad -\Delta u/\lambda \leq \ln\frac{\mu_0(\omega_1)}{\mu_0(\omega_2)} \leq \Delta u/\lambda \\ 1 & \text{if } \ln\frac{\mu_0(\omega_1)}{\mu_0(\omega_2)} > \Delta u/\lambda \\ 0 & \text{if } \ln\frac{\mu_0(\omega_1)}{\mu_0(\omega_2)} < -\Delta u/\lambda \end{cases} \tag{7}$$

Thus the posterior probability of choosing actions in different states can be obtained by combining equation 5 and 7. Since the information structure is independent of the $\mu \in supp(\gamma_0)$, private learning is only informative about Ω.

Note that, even in absence of social learning, the time $t = 0$ agents do not always learn perfectly about their types, hence the observed distribution of action contains both heterogeneities of idiosyncratic payoff and mistakes. Let $\epsilon_0^a = P(a, \omega_2|\gamma)$ and $\epsilon_0^b = P(b, \omega_1|\gamma)$ be the corresponding mismatch probabilities when choosing a and b at time $t = 0$ by type ω_2 and type ω_1 agents respectively. Since it is common knowledge that agents are Bayesian expected utility maximizer with the same cost of private learning, every agent chooses the same distribution of posteriors.

2.4 Time $t \geq 1$ agents

2.4.1 Optimal Learning Protocol

Any $t \geq 1$ period agent has two different choices for learning, namely social and private learning. The following lemma shows, optimal sequencing would always be of the form: *first social learning then private learning.*

Lemma 1. *Any agent in period $t \geqslant 1$ would optimally choose to learn socially first then privately.*

Proof. Suppose not. Consider an agent with belief γ_1 who chooses to learn privately first then observe n agents and update privately again if needed. I will show the agent can be made better off by choosing an alternate strategy where he first learns socially then privately. Since $\pi(s, \omega)$ is independent of μ the optimal choice of n would be the same under both strategies.

Let us first consider the case where private learning is optimal at every $\mu \in supp(\gamma_1)$ and after observing n. Suppose after observing n agents the belief is updated to $\tilde{\gamma}$. Then cost of private learning under the first strategy is

$$C_1 = \lambda \left[D(P(a|\mu)\,||P(a, \omega|\mu)) + E_{\gamma_1}(D(\tilde{P}(a, \omega|\tilde{\mu})||P(a, \omega|\tilde{\mu}))) \right]$$

where $\tilde{P}(a, \omega|\tilde{\mu}))$ is the updated posterior distribution of action upon observing n agents but need not be optimal at the updated belief $\tilde{\gamma}$, and $P(a, \omega|\tilde{\mu})$ be the optimal choice at $\tilde{\mu}$.

Under the second strategy the cost is

$$\begin{aligned} C_2 &= \lambda \left[E_{\gamma_1} \left[D(P\left(a|\tilde{\mu}\right)||P\left(a,\omega|\tilde{\mu}\right)) \right] \right] \\ &= \lambda \left[E_{\gamma_1} \left[D(P\left(a|\tilde{\mu}\right)||\tilde{P}\left(a,\omega|\tilde{\mu}\right)) + D(\tilde{P}(a,\omega|\tilde{\mu})||P(a,\omega|\tilde{\mu})) \right] \right]. \end{aligned}$$

The second equality is obtained by adding and substracting $E_{\gamma_1} \sum_{\omega\in\Omega} H(\tilde{P}(a,\omega|\tilde{\mu}))$ Since the agents are Bayesian, the order of private and social learning does not affect the distribution of belief generated. Thus for every $\mu \in supp(\tilde{\gamma})$ we have,

$$\begin{aligned} D(P\left(a|\mu\right)||P\left(a,\omega|\mu\right)) = D(P\left(a|\tilde{\mu}\right)||P\left(a,\omega|\tilde{\mu}\right)) \\ \Rightarrow C_1 = C_2. \end{aligned}$$

So the agent will be indifferent between the two strategies. If however, $\tilde{\gamma}$ generates a belief μ such that

$$\ln\frac{\mu\left(\omega_1\right)}{\mu\left(\omega_2\right)} \in (-\infty, -\Delta u/\lambda) \cup (\Delta u/\lambda, \infty) \quad \text{but} \quad \ln\frac{\mu_1\left(\omega_1\right)}{\mu_1\left(\omega_2\right)} \in [-\Delta u/\lambda, \Delta u/\lambda],$$

then the second strategy makes him strictly better off by saving the cost of private learning. Hence, proved. □

The main intuition of the proof is as follows: social learning is informative about the distribution of types and private learning is informative about the idiosyncratic type. Learning socially first changes the prior belief over DM's type, reducing the cost of learning. But if some social learning is done after private learning then there is a positive probability of paying an additional cost for private learning that could have been avoided ex-post. Since the cost of learning is a sunk cost, it is weakly better to learn socially first, privately later.

Let $\gamma_{x_n} \in \Delta(\Gamma)$ be the interim belief after observing x agents choosing a out of n randomly observed agents. The agent's problem becomes,

$$W\left(A,\gamma\right) = \max_n \sum_{\mu\in supp(\gamma_{x_n})} V\left(A,\mu\right)\gamma_{x_n}(\mu) - c\left(n\right) \tag{8}$$

2.4.2 Social learning and order of beliefs

Suppose an agent i at time t observes n agents from generation $t-1$, then he will update his belief over $\Delta\left(\Gamma\right)$ via Bayes rule. If the agent's observed sample is x_n, i.e. x out of n agents chose action a then the posterior probability of any distribution $\mu \in supp(\gamma_0)$ is,

$$P\left(\mu|\gamma, x_n\right) = \frac{P\left(x_n|\mu\right) P\left(\mu|\gamma\right)}{\int_{\nu\in supp(\gamma)} P\left(x_n|\nu\right) P\left(\nu|\gamma\right)} \tag{9}$$

and zero otherwise.

To calculate the $P(x_n|\mu)$, the agent needs to know the posterior choice probabilities of the earlier generation. This probability would be different for different generations. A time $t = 1$ agent knows any $t = 0$ agent had done only private learning. Given the common prior γ_0 and the marginal cost of learning λ the posterior choice probabilities can be obtained. All later generations need to take into consideration the optimal level of social learning in the earlier generation (n^*_{t-1}), distribution of interim beliefs over $\gamma(n^*_{t-1})$, and the distribution of posterior choice probability for each such distribution.

For a $t = 1$ agent, the probability of observing x_n, given prior μ would be,

$$P(x_n|\mu) = \sum_{k=0}^{n} \sum_{j=k^*}^{k^{**}} \binom{n}{x_n - 2j + k} \mu^{x_n-2j+k} (\epsilon_0^a)^j \left(1 - \epsilon_0^b\right)^{x_n-j} (1-\mu)^{n-x_n-k+2j} \left(\epsilon_0^b\right)^{k-j} (1-\epsilon_0^a)^{n-x_n-k+j} \tag{10}$$

where

$$k^* = \begin{cases} 0 & \text{if} \quad k < \min\{x_n, n - x_n\} \text{ or } \quad x_n \le k < n - x_n \\ k - n + x_n & \text{if} \quad k \ge \max\{x_n, n - x_n\} \text{ or } \quad n - x_n \le k < x_n \end{cases}$$

and

$$k^{**} = \begin{cases} k & \text{if} \quad k \le \min\{x_n, n - x_n\} \text{ or } \quad n - x_n \le k < x_n \\ x_n & \text{if} \quad k > \max\{x_n, n - x_n\} \text{ or } \quad x_n \le k \le n - x_n \end{cases}$$

Plugging the value obtained from equation 10 into equation 9 we can calculate $P(\mu|\gamma, x_n)$ for every $\mu \in \gamma$, and can update the belief to γ'_{x_n}.

Agents are ex-ante identical, any agent in a generation would choose the same n. Consider a period t agent who knows that period $t-1$ agents optimally chose to observe m people from generation $t-2$. Let X_m denote all possible sample distribution for sample size m. Let the mismatch probabilities be $\epsilon^a_{x_m,t} = P(a, \omega_2|\gamma, x_m, t)$ and $\epsilon^b_{x_m,t} = P(b, \omega_1|\gamma, x_m, t)$ after observing $x_m \in X_m$ in period t. Using prior γ_0, the distribution over X_m can be obtained, which generates implied distributions f^a_γ and f^b_γ over $\epsilon^a_{x_m,t}$ and $\epsilon^b_{x_m,t}$ respectively. Let $\epsilon^a_{m,t}$ and $\epsilon^b_{m,t}$ as $\epsilon^i_{m,t} = \int_{x_m \in X_m} \epsilon^i_{x_m,t} df^i_\gamma$, i.e., the expected probability of making mistake by choosing i in period $t-1$ after observing m many agents from generation $t-2$.

Since $t = 0$ agents choose only private learning, n^*_1 is common knowledge and iterating the argument and using the fact that all agents are ex-ante identical, given $c(n)$ and λ the sequence of optimal choice of n^*_t and the corresponding $\epsilon^i_{n^*_t}$ for $i \in A$ would also be common knowledge to all generations. $P(x_n|\mu)$ can thus be calculated by replacing ϵ^i_0 by $\epsilon^i_{n^*_{t-1},t}$ in equation 10 for any $t \ge 1$ generations.

2.4.3 Private Learning

Given lemma 1, we know agents first learn socially then with the updated belief γ'_{x_n} they learn privately. Following equation 5 the optimal private learning of an agent in any period $t \ge 1$ would be same as a $t=0$ agent, except with a different interim belief over Γ,

$$P\left(i,\omega|\gamma_{x_n}\right) = \frac{P\left(i|\gamma_{x_n}\right)e^{\frac{u(i,\omega)}{\lambda}}}{\sum_{j\in A}P\left(j|\gamma_{x_n}\right)e^{\frac{u(j,\omega)}{\lambda}}} \quad \forall i\in A \tag{11}$$

Note that the γ_{x_n} doesn't have a time dimension because the only way different generation would be different in their behavior is through social learning and γ_{x_n} captures the differences via social learning. Hence the agent with belief γ_{x_n} chooses to learn privately only if, $-\Delta u/\lambda \leq \ln\frac{\mu_{x_n}(\omega_1)}{\mu_{x_n}(\omega_2)} \leq \Delta u/\lambda$ where $\mu_{x_n}(\omega) = \sum_{\mu\in supp(\gamma_{x_n})}\gamma_{x_n}(\mu)\mu(\omega)$. For any other value of γ_{x_n} he would choose one action for sure.

3 Results

3.1 Optimal Learning Strategy

The following theorem characterizes the relationship between optimal private and social learning obtained from solving the optimization problem in equation 8.

Theorem 1. *Given the social learning cost function in 3 and the prior γ_0, there exist $0<\lambda^*<\lambda^j<\lambda^{**}<\infty$, such that*

1. *For all $\lambda\leqslant\lambda^*$, the optimal level of social learning at any period $t\geqslant 1$, $n_t^*\left(\lambda_1\right)\leqslant n_t^*\left(\lambda_2\right)$, where $\lambda_1\leqslant\lambda_2$, i.e. optimal social learning is non-decreasing in marginal cost of private learning or social and private learning are "substitutes".*

2. *For all $\lambda\in\left[\lambda^*,\lambda^j\right)\cup\left(\lambda^j,\lambda^{**}\right]$, the optimal level of social learning at any period $t\geqslant 1$, $n_t^*\left(\lambda_1\right)\geqslant n_t^*\left(\lambda_2\right)$ where $\lambda_1\leqslant\lambda_2$ and either $\lambda_1,\lambda_2\in\left[\lambda^*,\lambda^j\right)$ or $\lambda_1,\lambda_2\in\left(\lambda^j,\lambda^{**}\right]$, i.e optimal social learning is non-increasing in marginal cost of private learning or social and private learning are "complements".*

3. *For any $t\geqslant 1$, $\lim_{\lambda^j-}n_t^*\left(\lambda\right)<\lim_{\lambda^j+}n_t^*\left(\lambda\right)$, i.e. the optimal n_t^* takes an upward jump at λ^j.*

4. *For all $\lambda>\lambda^{**}$, no learning is optimal.*

The proof of theorem 1 is given in the appendix. The main intuition behind the result is as follows: the cost of private learning has two opposing effects on the optimal learning strategy. When the cost of private learning is high, DM would choose to substitute private learning by relatively cheaper social learning but a lower level of private learning implies the informativeness from social learning is also lower, since the predecessors have not chosen a high level of private learning as well.

When the cost of private learning is sufficiently low, the effect on loss of informativeness is relatively small, since everyone in the economy already chooses a high level of private learning. As the cost of private learning increases the substitutability component becomes relatively smaller, making the two types of learning complementary.

Futhermore, the shape of the private cost of learning makes the interim value function, after social learning, is given by figure 2. The optimal learning strategy is to avoid being in the decreasing section of the value function. This generates the jump in the learning strategy.

Finally, for high enough private cost of learning, all forms of learning become uninformative since early generations makes decision based on the prior belief alone. Thus all the following generations would choose no social learning as well, stopping all forms of learning in the economy.

3.2 Herding

Herding, as defined in the literature, refers to a phenomenon where, if sufficiently many agents have chosen the same action, all subsequent generations choose the same action ignoring their private signal. In this framework, we define herding as follows: for any prior γ_0 there exists $n_h(\gamma_0)$, such that if more than $n_h(\gamma_0)$ agent chooses action a (or b) in period t then agents in all periods $s \geq t$ would observe $n_s^* > 0$ agents and choose a (or b) without any private learning.

Corollary 1. *For all $\lambda \in [0, \infty] \setminus \left[\lambda^j, \lambda^{**}\right]$ there doesn't exist any herding, almost surely.*

Proof. Using theorem 1, for all $\lambda > \lambda^{**}$, the optimal social learning is zero. Hence herding cannot occur.

For all $\lambda < \lambda^j$, the $\bar{\mu}_n \in R_1$, hence for any belief μ_{x_n} after observing x_n $\ln \frac{\mu_{x_n}(\omega_1)}{\mu_{x_n}(\omega_2)} \in (-\Delta u/\lambda, \Delta u/\lambda)$. This implies for all possible values of x_n the optimal level of private learning is not zero. Since private learning is informative about idiosyncratic state, the probability of everyone choosing action a (or b) after private learning would be zero for any $\mu^* \in int\,(\gamma)$. Hence, there is no herding with probability 1.

But for $\lambda \in \left[\lambda^j, \lambda^{**}\right]$, $\bar{\mu}_n \in R_3$, hence there exists x^* such that if more than x^* agents choose a (or b), $\ln \frac{\mu(\omega_1)}{\mu(\omega_2)} \in (-\infty, -\Delta u/\lambda) \cup (\Delta u/\lambda, \infty)$. If sufficiently many agents chooses a (or b) in period t such that for all $t+1$ agents for all possible values of observed x_n, $\mu_{x_n} \in R_3$ then all agents in $t+1$ would choose a (or b) without any private learning. Then for every $s > t+1$ observed $x_n = n$ (or 0) and they would choose a without any private learning. Thus, herding cannot be ruled out. □

Consider the policy where the first few generations are not allowed to learn socially. This is a welfare-improving policy in the herding literature. But given 1 since herding is only optimal when $\lambda \in [\lambda^j, \lambda^{**}]$ where the two types of learning are complements, reducing social learning would reduce the level of private learning and can reduce the net expected payoff of the agent. Thus this is not unambiguously a welfare-improving policy. However, since for any μ, $V(\mu, \lambda_1) > V(\mu, \lambda_2)$ if $\lambda_1 < \lambda_2$, reducing λ increases expected payoff unambiguously.

4 Conclusion

For simplification, I have made several assumptions about learning protocols. Assuming the payoff only depends on idiosyncratic states implies social learning is only informative about the distribution of types. The assumptions of the homogeneous private and social cost of learning allow the agents in a later generation to update their information upon social learning from previous generations. Also, for social learning, we assumed the protocol of block learning where n is chosen before any observation. This assumption generates lemma 1.

In the appendix, we relax all these assumptions. We consider four extensions, namely, aggregate state affecting payoff, heterogenous cost of private and social learning, and sequential learning protocol. Under suitable adjustments, all these extensions preserve the main result of the paper.

The specific form of private learning function can be restrictive. For example, the cost of learning cannot have a fixed component. However, the mutual entropy function captures much relevant learning technology (e.g., cost increases in the precision of signals).

To conclude, this paper solves a model of individual stochastic choice where agents are rationally inattentive and face a costly social learning function. The optimal choice of social learning is non-monotonic in the marginal cost of private learning. Herding can only happen for an intermediate level of private cost of learning where the two types of learning are complements. The only unambiguously welfare-improving policy is to lower the marginal cost of private learning.

Bibliography

Banerjee, A. V. (1992). A simple model of herd behavior. *The Quarterly Journal of Economics*, 797–817.

Bikhchandani, S., D. Hirshleifer, and I. Welch (1992). A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of political Economy*, 992–1026.

Blackwell, D. et al. (1953). Equivalent comparisons of experiments. *The annals of mathematical statistics 24*(2), 265–272.

Brenner, T. (2006). Agent learning representation: advice on modelling economic learning. *Handbook of computational economics 2*, 895–947.

Caplin, A. and M. Dean (2015). Revealed preference, rational inattention, and costly information acquisition. *The American Economic Review 105*(7), 2183–2203.

Caplin, A., M. Dean, and J. Leahy (2019). Rational inattention, optimal consideration sets, and stochastic choice. *The Review of Economic Studies 86*(3), 1061–1094.

Caplin, A., J. Leahy, and F. Matějka (2015). Social learning and selective attention. Technical report, National Bureau of Economic Research.

Cover, T. M. and J. A. Thomas (2012). *Elements of information theory.* John Wiley & Sons.

Crémer, J. (1982). A simple proof of blackwell's "comparison of experiments" theorem. *Journal of Economic Theory 27*(2), 439–443.

Fernández, R. (2013). Cultural change as learning: The evolution of female labor force participation over a century. *The American Economic Review 103*(1), 472–500.

Fogli, A. and L. Veldkamp (2011). Nature or nurture? learning and the geography of female labor force participation. *Econometrica 79*(4), 1103–1138.

Kamenica, E. and M. Gentzkow (2011). Bayesian persuasion. *The American Economic Review 101*(6), 2590–2615.

Matějka, F. and A. McKay (2015). Rational inattention to discrete choices: A new foundation for the multinomial logit model. *American Economic Review 105*(1), 272–98.

Park, A. and H. Sabourian (2011). Herding and contrarian behavior in financial markets. *Econometrica 79*(4), 973–1026.

Sims, C. A. (2003). Implications of rational inattention. *Journal of Monetary Economics 50*(3), 665 – 690. Swiss National Bank/Study Center Gerzensee Conference on Monetary Policy under Incomplete Information.

Van Nieuwerburgh, S. and L. Veldkamp (2009). Information immobility and the home bias puzzle. *The Journal of Finance 64*(3), 1187–1215.

Woodford, M. (2012). Inattentive valuation and reference-dependent choice. *Unpublished Manuscript, Columbia University*.

A Appendix

A.1 Proof of Theorem 1

Proof. Given lemma 1, we can solve the optimization problem backward. First, for any intermediate belief $\mu \in supp(\gamma)$ the optimal private learning generate $V(\mu)$, then given $V(\mu)$, n^* is chosen to maximize $W(\gamma)$.

Step 1: No learning above λ^{**} Given γ_0 if $\ln\frac{\mu(\omega_1)}{\mu(\omega_2)} \in (-\infty, -\Delta u/\lambda) \cup (\Delta u/\lambda, \infty)$ then private learning is not optimal for any $t \geq 0$ and hence social learning is not informative, because agents choose according to their common prior. Consider $\lambda^{**} = \max\left\{-\ln\frac{\mu(\omega_1)}{\mu(\omega_2)}/\Delta u, \ln\frac{\mu(\omega_1)}{\mu(\omega_2)}/\Delta u\right\}$, then no learning is optimal for $\lambda > \lambda^{**}$.

Step 2: Shape of $V(\mu)$ Let $\mu = \sum_\mu \gamma(\mu)\mu(\omega_1)$, $p_a = P(a|\mu)$ and $\bar{\lambda} = \exp(\bar{u}/\lambda)$, $\underline{\lambda} = \exp(\underline{u}/\lambda)$. Substituting in equation 5 we get

$$V(\mu) = \bar{u}\left[\frac{\mu p_a \bar{\lambda} + (1-\mu)(1-p_a)\bar{\lambda}}{p_a \bar{\lambda} + (1-p_a)\underline{\lambda}}\right] + \underline{u}\left[\frac{(1-\mu)p_a \underline{\lambda} + \mu(1-p_1)\underline{\lambda}}{(1-p_a)\bar{\lambda} + p_a \underline{\lambda}}\right]$$
$$-\lambda\left[\mu\left\{\frac{p_a\bar{\lambda}}{p_a\bar{\lambda}+(1-p_a)\underline{\lambda}}\ln\frac{p_a\bar{\lambda}}{p_a\bar{\lambda}+(1-p_a)\underline{\lambda}} + \frac{(1-p_a)\underline{\lambda}}{p_a\bar{\lambda}+(1-p_a)\underline{\lambda}}\ln\frac{(1-p_a)\underline{\lambda}}{p_a\bar{\lambda}+(1-p_a)\underline{\lambda}}\right\}\right.$$
$$(1-\mu)\left\{\frac{p_a\underline{\lambda}}{(1-p_a)\bar{\lambda}+p_a\underline{\lambda}}\log\frac{p_a\underline{\lambda}}{(1-p_a)\bar{\lambda}+p_a\underline{\lambda}} + \frac{(1-p_a)\bar{\lambda}}{(1-p_a)\bar{\lambda}+p_a\underline{\lambda}}\ln\frac{(1-p_a)\bar{\lambda}}{(1-p_a)\bar{\lambda}+p_a\underline{\lambda}}\right\}$$
$$\left. -p_a\log p_a - (1-p_a)\ln(1-p_a)\right]. \quad (12)$$

Given λ, $V(\mu)$ is continuous in μ for $\mu \in [0,1]$ and continuously differentiable wrt μ in the open set $(0,1) \cap \left\{\frac{\underline{\lambda}}{\underline{\lambda}+\bar{\lambda}}, \frac{\bar{\lambda}}{\underline{\lambda}+\bar{\lambda}}\right\}^C$. Since

$$p(a) = \begin{cases} 1 & \text{if} \quad \frac{\bar{\lambda}}{\underline{\lambda}+\bar{\lambda}} < \mu \leq 1 \\ 0 & \text{if} \quad 0 \leq \mu \leq \frac{\underline{\lambda}}{\bar{\lambda}+\underline{\lambda}} \end{cases} \quad \Rightarrow V'_\mu = \begin{cases} \Delta u > 0 & \text{if} \quad \frac{\bar{\lambda}}{\underline{\lambda}+\bar{\lambda}} \leq \mu < 1 \\ -\Delta u < 0 & \text{if} \quad 0 < \mu \leq \frac{\underline{\lambda}}{\bar{\lambda}+\underline{\lambda}}. \end{cases}$$

The cutoffs are differentiable in λ, $\frac{d\frac{\bar{\lambda}}{\bar{\lambda}+\underline{\lambda}}}{d\lambda} = -\frac{d\frac{\underline{\lambda}}{\bar{\lambda}+\underline{\lambda}}}{d\lambda} = -\frac{(\bar{u}-\underline{u})}{\lambda^2}\frac{\bar{\lambda}\underline{\lambda}}{\left(\bar{\lambda}+\underline{\lambda}\right)^2} < 0$ hence, $\frac{\bar{\lambda}}{\underline{\lambda}+\bar{\lambda}}(\frac{\underline{\lambda}}{\underline{\lambda}+\bar{\lambda}})$ is decreasing(increasing) in λ. In the limit when $\lambda \to \infty$ the value function $V(\mu)$ becomes piecewise linear in $[0,1]$ with a kink at $1/2$.

In the region, $\left(\frac{\underline{\lambda}}{\underline{\lambda}+\bar{\lambda}}\frac{\bar{\lambda}}{\underline{\lambda}+\bar{\lambda}}\right)$ $p_a \in (0,1)$,

$$V'_\mu =$$
$$\underbrace{\frac{p_a\bar{\lambda}}{p_a\bar{\lambda}+(1-p_a)\underline{\lambda}}\left[\bar{u}-\lambda\log\frac{p_a\bar{\lambda}}{p_a\bar{\lambda}+(1-p_a)\underline{\lambda}}\right]}_{(1)} - \underbrace{\frac{(1-p_a)\bar{\lambda}}{(1-p_a)\bar{\lambda}+p_a\underline{\lambda}}\left[\bar{u}-\lambda\log\frac{(1-p_a)\bar{\lambda}}{(1-p_a)\bar{\lambda}+p_a\underline{\lambda}}\right]}_{(2)}$$
$$-\underbrace{\frac{p_a\underline{\lambda}}{(1-p_a)\bar{\lambda}+p_a\underline{\lambda}}\left[\underline{u}-\lambda\log\frac{p_a\underline{\lambda}}{(1-p_a)\bar{\lambda}+p_a\underline{\lambda}}\right]}_{(3)} + \underbrace{\frac{(1-p_a)\underline{\lambda}}{p_a\bar{\lambda}+(1-p_a)\underline{\lambda}}\left[\underline{u}-\lambda\log\frac{(1-p_a)\underline{\lambda}}{p_a\bar{\lambda}+(1-p_a)\underline{\lambda}}\right]}_{(4)}$$
$$+\mu\underbrace{\frac{\bar{\lambda}\underline{\lambda}}{\left(p_a\bar{\lambda}+(1-p_a)\underline{\lambda}\right)^2}\frac{\bar{\lambda}+\underline{\lambda}}{\bar{\lambda}-\underline{\lambda}}\left[\bar{u}-\underline{u}-\lambda\log\frac{p_a\bar{\lambda}}{(1-p_a)\underline{\lambda}}\right]}_{(5)}$$
$$\underbrace{-(1-\mu)\frac{\bar{\lambda}\underline{\lambda}}{\left(p_a\underline{\lambda}+(1-p_a)\bar{\lambda}\right)^2}\frac{\bar{\lambda}+\underline{\lambda}}{\bar{\lambda}-\underline{\lambda}}\left[\bar{u}-\underline{u}-\lambda\log\frac{(1-p_a)\bar{\lambda}}{p_a\underline{\lambda}}\right]}_{(6)} - \underbrace{\lambda\frac{\bar{\lambda}+\underline{\lambda}}{\bar{\lambda}-\underline{\lambda}}\log\frac{1-p_a}{p_a}}_{(7)}. \quad (13)$$

Since the value function is symmetric in μ around $\mu = 1/2$ consider only $\mu \geq 1/2$ region.

Rearranging terms and plugging p_a (refer 7),

$$(5)-(6)-(7)=\underbrace{\lambda\frac{\bar{\lambda}+\underline{\lambda}}{\bar{\lambda}-\underline{\lambda}}\log\frac{p_a}{1-p_a}}_{\substack{\geq 0 \quad \text{if} \quad \mu\geq 1/2\\ \leq 0 \quad \text{if} \quad \mu\leq 1/2}}\underbrace{1-\frac{\bar{\lambda}\underline{\lambda}}{\left(\bar{\lambda}+\underline{\lambda}\right)^2}\left(\frac{1}{\mu}+\frac{1}{1-\mu}\right)}_{\substack{\geq 0 \text{ if } 1/2\leqslant\mu<\bar{\lambda}/\bar{\lambda}+\underline{\lambda}\\ \leq 0 \text{ if } \underline{\lambda}/\bar{\lambda}+\underline{\lambda}<\mu\geqslant 1/2}}$$

$$((1)+(4))-((2)+(3))=\underbrace{\frac{\bar{\lambda}\underline{\lambda}\left(2p_a-1\right)\left(\bar{u}-\underline{u}\right)}{\left(p_a\bar{\lambda}+\left(1-p_a\right)\underline{\lambda}\right)\left(p_a\underline{\lambda}+\left(1-p_a\right)\bar{\lambda}\right)}}_{(1')}+\underbrace{2\lambda}_{>0}$$

$$\underbrace{+(\bar{u}-\underline{u})\frac{\bar{\lambda}\left(p_a^2+\left(1-p_a\right)^2\right)+2p_a\left(1-p_a\right)\underline{\lambda}}{\left(p_a\bar{\lambda}+\left(1-p_a\right)\underline{\lambda}\right)\left(p_a\underline{\lambda}+\left(1-p_a\right)\bar{\lambda}\right)}}_{>0}-\underbrace{\frac{\bar{\lambda}\underline{\lambda}\left(2p_a-1\right)}{\left(p_a\bar{\lambda}+\left(1-p_a\right)\underline{\lambda}\right)\left(p_a\underline{\lambda}+\left(1-p_a\right)\bar{\lambda}\right)}\log\frac{p_a}{1-p_a}}_{(2')}.$$

Rearranging further,

$$\left(1'\right)-\left(2'\right)=\underbrace{\frac{\bar{\lambda}\underline{\lambda}\left(2p_a-1\right)}{\left(p_a\bar{\lambda}+\left(1-p_a\right)\underline{\lambda}\right)\left(p_a\underline{\lambda}+\left(1-p_a\right)\bar{\lambda}\right)}}_{\geqslant 0 \text{ for } \mu\geqslant 1/2}\lambda\log\frac{\left(1-\mu\right)\frac{\bar{\lambda}}{\underline{\lambda}}-\mu}{\mu-\left(1-\mu\right)\frac{\underline{\lambda}}{\bar{\lambda}}}.$$

For $\mu\geq 1/2$,

$$\left(1'\right)-\left(2'\right)=\begin{cases}\leq 0 & \text{if } \mu\geq\frac{\bar{\lambda}^2+\underline{\lambda}^2}{\left(\bar{\lambda}+\underline{\lambda}\right)^2}\\ \geq 0 & \text{otherwise}\end{cases}$$

Combining we get,

$$\lim_{\epsilon\to 0}V'\left(\mu=\frac{\bar{\lambda}}{\bar{\lambda}+\underline{\lambda}}-\epsilon\right)=\underbrace{-\lim_{\epsilon\to 0}\epsilon\frac{\bar{u}-\underline{u}}{\bar{\lambda}\underline{\lambda}\left(\bar{\lambda}-\underline{\lambda}\right)}\left(\bar{\lambda}^2+\underline{\lambda}^2\right)\left(\bar{\lambda}+\underline{\lambda}\right)}_{(=0)}-\underbrace{\lambda\frac{\bar{\lambda}+\underline{\lambda}}{\bar{\lambda}-\underline{\lambda}}\log\frac{\bar{\lambda}}{\underline{\lambda}}}_{(<0)}<0$$

using the convention $0\log 0=0$. Also, $\lim_{\mu\to 1/2^+}V'(\mu)\downarrow 0$. Since $V(\mu)$ is continuous in μ by intermediate value theorem there exists an unique $\mu_{max}\in\left(\frac{\bar{\lambda}^2+\underline{\lambda}^2}{\left(\bar{\lambda}+\underline{\lambda}\right)^2},\frac{\bar{\lambda}}{\bar{\lambda}+\underline{\lambda}}\right)$ where $V(\mu)$ attains an interior maximum.

Since $\lim_{\mu\to\frac{\bar{\lambda}}{\bar{\lambda}+\underline{\lambda}}^-}V(\mu)<0$ and $\lim_{\mu\to\frac{\bar{\lambda}}{\bar{\lambda}+\underline{\lambda}}^+}V(\mu)>0$, $V(\mu)$ obtains a minima at $\frac{\bar{\lambda}}{\bar{\lambda}+\underline{\lambda}}$. Considering $\mu\leq 1/2$, we get $V(\mu)$ obtains another minima at $\mu=1/2$ where

$$V\left(\mu = 1/2\right) < V\left(\mu = \frac{\bar{\lambda}}{\bar{\lambda} + \underline{\lambda}}\right)$$

i.e., V attains global minima at $\mu = 1/2$. Figure 1 illustrates the shape of the value function for different λs.

As λ increases, since

$$\frac{d\frac{\bar{\lambda}^2+\underline{\lambda}^2}{\left(\bar{\lambda}+\underline{\lambda}\right)^2}}{d\lambda} = \frac{\bar{u}-\underline{u}}{\lambda^2}\left(\frac{\bar{\lambda}}{\underline{\lambda}}+1\right)\frac{\bar{\lambda}}{\underline{\lambda}}\left(1-\frac{\bar{\lambda}}{\underline{\lambda}}\right) < 0$$

and $\frac{d\frac{\bar{\lambda}}{\bar{\lambda}+\underline{\lambda}}}{d\lambda} < 0$ then the maximizer μ_{max} decreases with λ. As the interval in which μ_{max} lies shifts to the left towards 1/2.

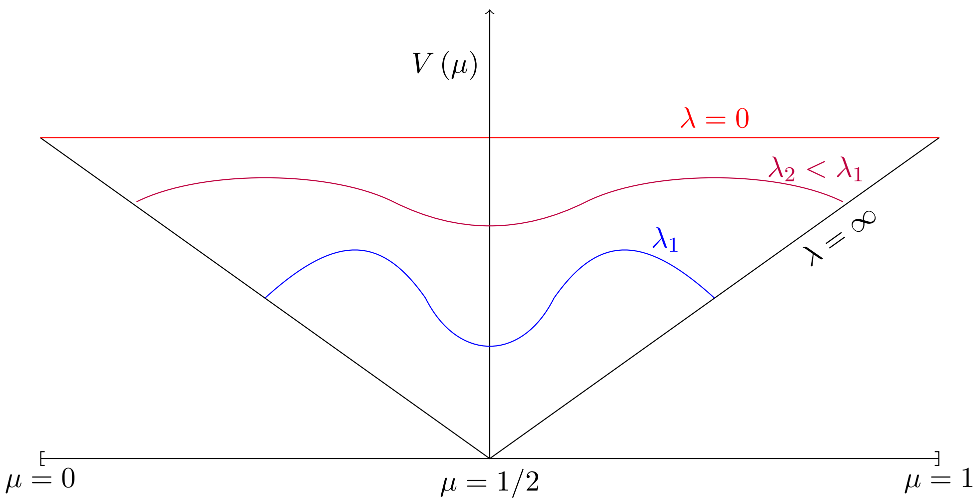

Figure 1: The value function $V(\mu)$ for different λs

Step 3: Interaction between private and social learning Note that the probability of mismatch increases with λ, i.e., $\frac{d\left(\epsilon^a+\epsilon^b\right)}{d\lambda} > 0$. This generates a bound on updated belief after social learning. Given a common prior γ_0 let us define $\bar{\mu}_n$ as the most exterme (all a or all b) posterior belief after observing n agents. WLOG let $\bar{\mu}_n > 1/2$. Fig 2 shows the three possible regions where $\bar{\mu}_n$ can lie. I want to show no agent will choose n such that $\bar{\mu}_n \in R_2 = (\mu_{max}, \mu^h)$ where $V(\mu^h) = V(\mu_{max})$.

Suppose, the agent choose n_1 such that $\bar{\mu}_{n_1} \in R_2$. Consider an alternate strategy of choosing $n_2 < n_1$ such that given λ, n_2 maximizes $\bar{\mu}$, given $\bar{\mu}_{n_2} \leq \mu_{max}$. Since $n_2 < n_1$, for any $x \in [0, n_2]$ $\mu_{x_{n_2}}$ is closer to μ_0 than $\mu_{x_{n_1}}$. For example, if $\mu_0 > 1/2$, observing 4 out of 7 agents choosing a is a weaker evidence for a higher μ than 4 out of 5 agents choosing a. Thus $V(\mu_{x_{n_1}}) \leq V(\mu_{x_{n_2}})$ since

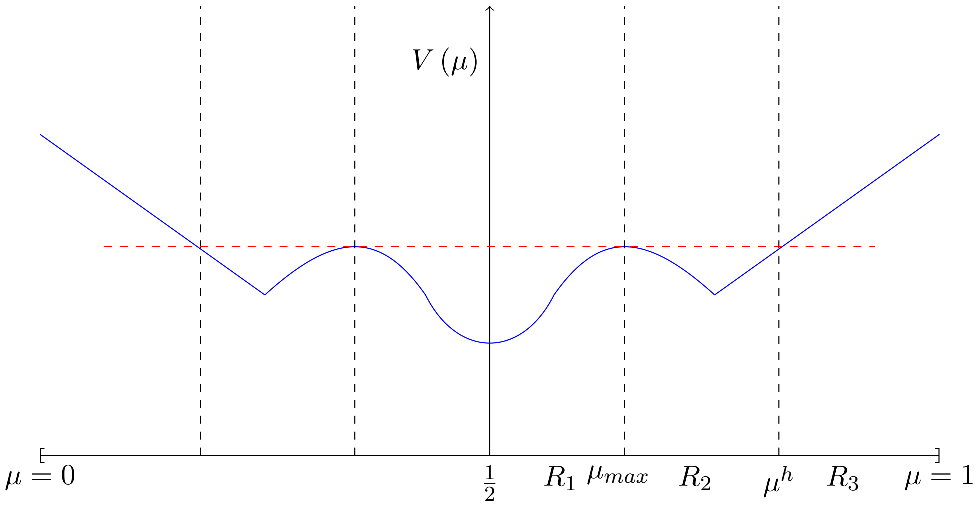

Figure 2: Three regions: $R_1 = [\frac{1}{2}, \mu_{max}]$, $R_2 = (\mu_{max}, \mu^h)$ and $R_3 = [\mu^h, 1]$

both lie below μ_{max}. For any $x \in [n_2, n_1]$, $V(\bar{\mu}_{n_2}) \geq V(\mu_{x_{n_1}})$ since $\mu_{x_{n_1}} \in R_2$ and $\bar{\mu}_{n_2} \in R_1$. Thus

$$\sum_{x=1}^{n_1} V(\mu_x)\gamma(\mu_x) \leq \sum_{x=1}^{n_2} V(\mu_x)\gamma(\mu_x).$$

Since $c(n_1) > c(n_2)$, the agent would be better off by choosing n_2. Hence optimal n would be such that $\bar{\mu}_n \in R_1 \cup R_3$.

Note, μ_{max}, $\frac{\bar{\lambda}}{\bar{\lambda}+\underline{\lambda}}$ is decreasing in λ and $\gamma(\mu|x_n, \lambda)$ is continuous in λ. This implies for any n, there exists $\lambda^j(n)$ such that $\forall\lambda < \lambda^j(n)$, $\bar{\mu}_n \in R_2$ but $\forall\lambda > \lambda^j(n)$ $\bar{\mu}_n \in R_3$. Let

$$\lambda' = \min_n \lambda^j(n).$$

Since $c(N) > \bar{u}$ and λ imposes restriction on belief updating by ϵ_i, $\lambda' < \infty$. For any $\lambda < \lambda^j$, it is optimal for the agent to choose n to remain in R_1 since $\bar{\mu}_{\bar{n}}(\lambda) \in R_2$ and for all $\lambda > \lambda'$ the agent may choose n such that $\bar{\mu}_n \in R_3$. Consider $\lambda^j \geq \lambda'$ such that

$$\sum_{i=1}^{n_1} V\left(\mu_i^{n_1}\right) Pr\left(\mu_{n_1}|\gamma\right) - c\left(n_1\right) = \sum_{i=1}^{\bar{n}} V\left(\mu_{\bar{n}}\right) Pr\left(\mu_{\bar{n}}|\gamma\right) - c\left(\bar{n}\right)$$

where n_1 be the maximum value of n such that $\bar{\mu}_n \in R_1$. At λ_j agent is indifferent between R_1 and r_3. Hence, for $\lambda > \lambda^j$ the optimal n^* would be such that $\bar{\mu}_{n^*} \in R_3$.

Step 4: Optimal n for $\lambda < \lambda^j$ A smaller λ implies social learning is more informative but it also makes private learning cheaper. When the first effect dominates, the two types of learning are complements, otherwise, they are substitutes.

Since $\bar{\mu}_{n^*} \in R_1$ for all $\lambda < \lambda_j$, $\lim_{\lambda\to 0^+} V'_{\mu\lambda} \geq 0$, because V_μ becomes steeper in R_1 with an increase in λ. Since $c\left(n\right)$ is same for all λ but a steeper V_μ implies larger increase in the benefit from an increase in μ, n^* would be non-decreasing λ for $\lambda \to 0$.

Since, μ_{max} is decreasing in λ, $\lim_{\lambda \to \lambda^j -} V'_{\mu\lambda} \leqslant 0$ for $\mu \in \mathcal{N}_\epsilon\left(\bar{\mu}^{n^*}\left(\lambda^j\right)\right)$, i.e n^* would be non-increasing in λ in left neighborhood of λ'_n.

As V'_μ is continuous in λ and $\bar{\mu}_n$ for all λ, by intermediate value theorem there exists a $\lambda^* \in \left[0, \lambda^j\right]$ such that n^* is non-decreasing for $\lambda \leqslant \lambda^*$ and non-increasing for $\lambda \geqslant \lambda^*$. This proves the part (i) of the theorem.

Step 5: Optimal n **for** $\lambda \in (\lambda^j, \lambda^{**})$ For $\lambda \in (\lambda^j, \lambda^{**})$, $\bar{\mu}_n \in R_3$. Since $\epsilon^a + \epsilon^b$ is increasing in λ, i.e. $\bar{\mu}_n$ is decreasing in λ given n. But the cost function is same for all λ implies n^* is non-increasing in λ. This concludes the proof of part (ii) of the theorem. □