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CONDORCET JURY THEOREM IN A SPATIAL MODEL OF ELECTIONS

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Condorcet Jury Theorem in a Spatial Model of Elections

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Abstract

Two alternatives P and Q are in contention. There are two states and each state identifies a location of P and Q on the unidimensional policy space. Voters have noisy information about the state. We provide a characterization of limit equilibrium outcomes as the electorate increases unboundedly. If P lies on the same side of Q in both states, then information aggregation is guaranteed. If P is to the right of Q in one state and to the left of Q in the other, then there are three distinct equilibrium sequences, only one of which is full information equivalent. This shows how distributional uncertainty leads to failure of information aggregation.

1 Introduction

Voters have noisy information about candidates and their policies. Can elections solve this informational problem and select the true majority preferred alternative? The celebrated Condorcet Jury Theorem (Condorcet (1785)), henceforth CJT, asserts that if *all voters have the same preference*, the electoral outcome under majority rule indeed corresponds to the best alternative. According to the CJT, elections are *full information equivalent* in the sense that the outcome corresponding to the aggregate of all private information in the electorate is selected (Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1997), Myerson (1998), Wit (1998), Duggan and Martinelli (2001), Meirowitz (2002)), Barelli et al (2020)). But in the real world, voters may disagree on their preferred alternatives even when they have the same information. In this paper, we examine if the CJT holds in the

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unidimentional spatial model (Downs, 1957), which is the standard model for analysing political competition with voters having diverse preferences.

In our model, voters have single-peaked preferences with their "ideal points" distributed on the left-right continuum which is taken to be the unit interval. The alternatives in contention (candidates or policies) are located as points on this policy space. We consider two alternatives $A \in \{P, Q\}$ and there is uncertainty regarding the exact location of these alternatives. A binary state variable $\omega \in \{L, R\}$ captures such uncertainty: the location of policy A in state ω is $x_\omega^A \in (0,1)$. In words, P and Q are located either at (x_L^P, x_L^Q) - state L or at (x_R^P, x_R^Q) - state R. We also assume that L is the "left state" and R is the "right state" in the sense that $x_L^P < x_R^P$ and $x_L^Q < x_R^Q$. Voters obtain noisy private sigmals regarding the state. In this environment, we ask if the electoral outcome coincides with the majority-preferred alternative under common knowledge of the state.

Our main result is that the property of Full Information Equivalence (henceforth, FIE) depends only on whether there is uncertainty in the locational order of alternatives. To explain this, suppose that $x_\omega^P < x_\omega^Q$ for $\omega \in \{L, R\}$. Here, P is known to lie to the left of Q in both states despite the uncertainty about their specific locations. We refer to this as the *ordered alternatives* environment. In such environments, all state-sensitive voters have the same ranking over alternatives, much like the canonical jury settings. The ex-post majority preferred alternative is guaranteed to be elected in this environment.

In contrast, suppose that $x_L^P < x_L^Q < x_R^Q < x_R^Q$. Here, P lies to the left of Q in state L but to the right of Q in state R. These settings - referred to as *unordered alternatives* environments - have two related features. First, there are two groups of state-sensitive voters with opposed preferences. Voters sufficiently to the right prefer P in state R and Q in state L, while the voters beyond a threshold in the left have opposite rankings in each state. Second, due to the two-state structure, there is a central alternative (Q in this case) in the sense that both its possible locations are flanked on two sides by the two possible locations of the rival alternative. Q has a "lower risk exposure" in the sense that for every voter, a change in state induces a larger change in utility from Q than for P. The alternative P receives votes from the relevant group only if the common belief indicates the corresponding state sufficiently strongly. The altertantive Q, on the other hand, has the advantage of being preferred by most voters for moderate beliefs and by moderate voters for most beliefs. Of course, beliefs are determined endogenously in equilibrium.

We look at sequences of type-symmetric Nash equilibria of the voting game as the number of voters grows unboundedly and compare the limit outcome with the full information outcome. The contribution in our work is to characterize the full set of equilibria (in the limit). There are essentially three equilibrium sequences for the unordered alternatives environment - information is aggregated in one of these. In the two others, the "centrally located alternative" wins in both states irrespective of whether it is majority preferred or not. Our results do not depend on the particular threshold voting rule, prior likelihoods of the states, the distribution of voter preferences, the size of voter groups with interest conflict or the extent of noise in private information.

According to our result, in an election between two candidates P and Q, CJT holds as long as it is known that P lies to the left (or right) of Q, even if their precise policy positions are uncertain. However, if one or both candidates are running on an platform of efficiency or of populism, it may be unclear whether the leftist or the rightist voters have a higher benefit from P relative to Q. A similar situation may arise if a lesser known challenger faces an incumbent and there is a "large" uncertainty about the challenger's ideological position relative to that of the incumbent. Later in this paper we discuss some empirical evidence that voters are often not able to order candidates correctly along salient issue dimensions. In these situations, the majority preferred candidate may lose the election.

State-contingent preference conflict often arises in settings of distributive politics when there is uncertainty about the outcome of a policy, translating into uncertainty over the identity of who gains and who loses from a policy change. This phenomenon is usually termed as distributional uncertainty (Gersbach 2000, Ali, Siga and Mihm 2018). Consider the example of a vote over trade liberalization adapted from Fernandez and Rodrik (1991). This could be an election where the major issue is trade reforms, or an actual referendum over joining (or leaving) an economic union with other countries as had happened in the United Kingdom in 2016. If the country has a two-sector economy, trade liberalization would cause the sector with comparative advantage to grow and the other to shrink. If there is an uncertainty about which sector the comparative advantage lies in, we have conflicting preferences between the two sectors. In other words, if it is uncertain whether the proposed trade reform will make voters employed in industry better off at the cost of those in agriculture or the other way round, the reform may be blocked even when ex-post it is actually favoured by the majority.

Such a situation can also arise when communities vote over financing lumpy public investments which benefit specific groups disproportionately. Suppose a city with two districts is voting to raise taxes to build a school, but the location of the school is not yet determined. Residents of the district where the school is finally located will gain in the net, while residents of the other district will lose from this increase in taxes. This is another situation where there are two groups with opposed preference in each state, and the project may get stalled.

This is not the first paper pointing out that information aggregation may be impeded if there are voter groups with opposed preferences (Kim and Fey (2008), Gersbach (1995), Ali, Mihm and Siga (2018)). Bhattacharya (2013) provides general conditions on co-monotonity between preferences and information for the existence of an inefficient equilibrium. While our environment is a special case of Bhattacharya (2013), the additional spatial structure allows us to characterize the entire set of limit equilibria. More importantly, the current paper tracks the behavior of swing voters and demonstrates why the minority preferred alternative may carry the election. Our message is that for an important class of threshold rules (those that induce different outcomes in different states under full information), efficient and inefficient equilibria co-exist. Thus, we view the fundamental problem in elections as endemic co-ordination uncertainty.

1.1 Structure of Equilibria

We assume that each voter takes into account the fact that his choice matters only when the others' votes are tied, and conditions his voting decision on this event (Austen-Smith and Banks 1996). Therefore, each equilibrum is characterized by a commonly shared belief β over states ($\beta = \Pr(\omega = R)$) such that (i) equilibrium strategy is individually optimal with respect to belief β and (ii) given that others are using the equilibrium strategy, the commonly inferred distribution over states conditioning on a tie is β. Since this belief acts like a common prior but is induced in equilibrium, we call it the "induced prior belief". In this paper, we identify the set of limit values of induced priors using a technique developed in Bhattacharya (2013). We employ a two-step procedure. First, we identify the expected vote share in each state as a function of the exogenous prior. Then we determine which values of the prior induce the same belief conditioning on a tie (as the electorate grows unboundedly). In other words, the exercise is that of finding the (limit of) fixed points in the space of beliefs. While we apply this methodology to the unidimensional spatial model, the same method can be used to find the set of equilibria for a much wider variety of environments.

Denote by $t(\omega, \beta)$ the expected vote share for P in state $\omega \in \{L, R\}$. This is obtained by considering the optimal decision of each type given belief β, and then integrating over types. The main part of our characterization is the following result: Any belief $\beta \in (0, 1)$ is a limit induced prior if and only it produces equal pivot probabilities in each state. For $\beta = \{0, 1\}$, this condition is necessary but not sufficient. To see why this is necessary, observe that if for some β the pivot probability in one state is larger, then all belief conditional on pivotality tends to be concentrated on that state as the electorate grows, implying that the corresponding β cannot be a fixed point.[1] This condition allows us to pin down limit values of the induced prior from the shape of the functions $t(\omega, \beta)$ in each environment.

For instance, consider the simple majority rule $\theta = \frac{1}{2}$. Equality of pivot probability holds if $t(L, \beta)$ and $t(R, \beta)$ are equidistant from $\frac{1}{2}$, for which we need either $t(L, \beta) + t(R, \beta) = 1$ or $t(L, \beta) = t(R, \beta)$. Finding equilibria boils down to checking beliefs which satisfy one of these two conditions. To complete the characterization of equilibria, we also have to check if $\beta = 0$ or $\beta = 1$ are fixed points (in the limit).

Consider the ordered alternatives case where P is known to lie to the left of Q (say, $x_L^P < x_R^P < x_L^Q < x_R^Q$). In this environment, there is an interval $[x_L, x_R]$ of independent types that prefer P in state R and Q in state L. Optimal behavior under uncertainty (given belief β) has the following structure: there is a responsive interval of types (subset of independents) vote P on r-signal and Q on l-signal, while those to the left (right) of the responsive set vote P (Q) for both signals. As β increases, more independents switch to P,

[1] Sufficiency of this result is proved constructively.

so the function $t(\omega, \cdot)$ is increasing in both states $\omega \in \{L, R\}$. Also, $t(L, \beta) < t(R, \beta)$ for all $\beta \in (0, 1)$ due to informative voting by responsive voters. As a result, there is a unique equilibrium induced prior β' satisfying $t(R, \beta') + t(R, \beta') = 1$. Since $t(L, \beta') < \frac{1}{2} < t(R, \beta')$, by the Law of Large numbers, P gets the majority in state R and Q in state L. This is also the full information equivalent outcome provided that there are enough independent voters. In this equilibrium, the responsive set of voters contains the median type. The description of moderate swing voters driving election results is consonant with the standard description of electoral behavior by journalists, academics and electoral commentators (see for example, The Swing Voter in the American Politics (2008), ed. William G. Mayer). However, when the alternatives are not ordered, voting behavior may not conform to this description.

Consider now an environment with $x_L^P < x_L^Q < x_R^Q < x_R^P$. Here, there are two groups of state-senstive voters: a set of types $[0, x_L]$ - the L-group, prefer P in state L and Q in state R, while a set $[x_R, 1]$ - the R-group have the exact opposite preference. The types in $[x_L, x_R]$ are Q-partisans. Assuming that the R-group consists of more than 50% voters, a majority prefers P in state R and Q in state L under full information. In this setting, P receives more votes either when the belief strongly indicates state L (low β) and most of the L-group votes for P or when β is very high and most of the R-group votes P. Thus, the vote share functions $t(L, \beta)$ and $t(R, \beta)$ are both U-shaped. Moreover, $t(L, \beta)$ and $t(R, \beta)$ cross exactly once at β^*. For lower beliefs the responsive voters are predominantly in the L-group and hence $t(L, \beta) > t(R, \beta)$ and for $\beta > \beta^*$, responsive voters are predominantly in the R-group implying $t(L, \beta) < t(R, \beta)$. Such shape of vote share functions leads to three fixed points in the space of beliefs (in the limit).

Observe that the median voter lies in the R-group. There is one equilibrium where the responsive set contains the median voter the full information outcome obtains almost surely in the limit.

There are two other equilibria in which Q is elected in both states. One such equilibrium occurs at β^*, the intersection of vote share functions. Since $t(L, \beta^*) = t(R, \beta^*) < \frac{1}{2}$, Q wins in both states almost surely. In this equilibrium, only the extremists at either end of the ideological spectrum are responsive to information. Since pivotality does not strongly indicate one state or the other, moderate voters in both groups vote for Q. We call this the *activist voting equilibrium* since only the types who have the largest utility difference between the alternatives vote for P, provided their signal suggests accordingly. To an outside observer, this voting behavior appears to be one where P is supported by a coalition of some voters from the far right and some from the far left.

Another equilibrium occurs with β converging to 0. Independent of their private information, everyone in the R-group votes for Q and almost everyone in the L-group votes for P. Since the size of L-group is less than 50%, P obtains too few votes in either state. With a vanishing fraction of voters in the L-group voting according to their signal, a tie (while itself very rare) is much more likely when these voters vote for P, i.e., when the state is 0. Since all voters condition their vote on this very event, almost everyone votes as if the state is *known* to be L. We describe this equilibrium as a *block voting equilibrium*, with opposed groups voting for opposite alternatives.

Finally, one might wonder if there is also an equilibrium with β approaching 1, where all R-voters vote for P. But this cannot be a limit value of equilibrium beliefs: for a sequence β^n approaching 1, $t(R, \beta^n) > t(L, \beta^n) > 50\%$ along the sequence, making a tie far more likely in state $\omega = L$.[2]

In the main section of the paper we consider only consequential rules. These are threshold voting rules with the property that under full information, P wins in one state and Q in another. With such rules, whether information is aggregated or not depends on equilibrium selection. In section 6, we consider all non-unanimous threshold rules and show that for a continuum of voting rules, there exists no equilibrium that aggregates information. In particular, these are rules for which P should win in both states under full information, but in each equilibrium Q wins almost surely in at least one state. We believe that this strong failure of information aggregation is of independent theoretical interest.

1.2 Applications

Our results suggest that in an environment where there is uncertainty about the identity about winners and losers from a change of alternatives, information may not be aggregated in elections (or large committees). We have already suggested a few environments which are subject to such distributional uncertainty. In this section, we elaborate on a few more.

An important class of applications arise if we consider that there is uncertainty about only one of the two alternatives. If $x_L^Q = x_R^Q = x^Q$, then we can think of the alternative Q as the *status quo* with known location, while P is a policy proposal that is put to electoral test against the status quo. There is a large literature in both economics and political science that equates policymaking with experimentation, making the point that choosing or electing a policy is rarely the same as ascertaining an outcome.[3] Our result suggests that when the policy under consideration creates distributional uncertainty, the elections have a status quo bias.

A direct application of our framework is in referenda, which are by definition single issue elections. In fact, some proponents of direct democracy invoke the Condorcet Jury Theorem in order to suggest that referenda aggregate information efficiently even if voters may be mistaken about the policy consequences (Matsusaka 2005, Lupia 2001).[4] This paper points out that the argument hinges on the nature of the issue on ballot. If it is an ideological issue like gay marriage or abortion, we are typically in the ordered alternatives

[2] With similar reasoning, in the ordered alternatives environment with majority rule we can rule out equilibrium beliefs approaching either 0 or 1.

[3] See Lindblom (1959) for an early enunciation of the idea of policymakers "muddling through" policies in search of good outcomes. A more recent example is Callander (2011).

[4] For example, Matsusaka (2005, p. 193) claims that "Direct democracy can be effective even when voters have no more or even worse information than legislators....aggregating the opinions of a million voters can be highly accurate by the Law of Large numbers even if each person's chance of being right is small (this is a version of Condorcet Jury Theorem..)"

case: it is clear whether the proposal is to the right or left of the status quo. However, if the issue on the anvil is distributional, i.e., trade or immigration reform, we are more likely to be in the unordered alternatives environment and the reform is no longer guaranteed to pass even if it is favored by the majority of citizens.

An instance of a referendum that would possibly fit our framework would be the "Brexit" referendum in the United Kingdom in June 2016. In an environment where trade continues to induce sweeping and unpredictable changes to the economy, prevention of trade retains the current and familiar economic structure. One would expect a bias towards autarky in such cases. The same idea can be applied to elections where a central issue is trade and immigration. Donald Trump's success based on his vehement anti-trade position in the American Presidential election of 2016 and good showing in the 2020 elections baffling all pre-poll predictions is consistent with the autarky bias predicted by our model.

In fact, there is some evidence that in case of referenda over trade or immigration reforms, the pattern of coalitions are indeed like the activist voting equilibrium that we identify. Johnston et al (1996) (see page 13 and references therein) argues that in various countries, the referendum to ratify the Maastricht Treaty (i.e., joining the European Union) was opposed by a coalition of the far left and far right. Among these countries, while the measure failed in Switzerland (1992), Norway (1994) and Denmark (1992 and 1994), it passed by a narrow majority in France (1992).

Status quo bias in referenda have been well documented in empirical work. In general, the details of the referendum process and the rules for passage vary, making comparison across countries or aggregation over instances difficult. In Australia, all amendments to the constitution are required to be passed via referenda in which voting is compulsory for everyone on the electoral roll. As of date, of the 44 proposals put forth for referendum in Australia, only 8 have passed. In Switzerland, the "gold standard" for direct democracy, only 36% of all optional referenda have passed in the period from 1991 till 2006, and authors have held direct democracy responsible for its slow growth during the nineties, delays in reforms and so on (Kirchgassner 2007, 2008). In the United States, the success rate of statewide ballot initiatives from 1904 till 2019 is about 41% [5], We provide a simple theoretical model to explain a less-than 50% success rate in terms of co-ordination failure among voter groups.

While we use the metaphor of referenda, the current paper applies to political races between two candidates or parties to the extent they can be reducible to a single, possibly ideological dimension. In case of high profile national elections, we often know which candidate is to the right and which one is to the left simply from their party identities. However, in may other situations voters are faced with an uncertainty over the order of the alternatives. In primaries where both candidates are from the same party, the left-right

[5] Of 2610 initiatives put on ballot since the first one in Oregon in July 1904, only 1080 have passed. Source: Historical database maintained by the Initiative and Referendum Institute at the University of Southern California (http://www.iandrinstitute.org/data.htm)

order of candidates may not be clear. In local or municipal elections, candidates often run on the plank of efficiency or local issues, making it difficult for voters to use party affiliation as an informational shortcut for candidate positions.

Even when the candidates do take clearly defined issue positions, there is substantial evidence to the effect that voters often fail to learn the positions or, worse still, fail to even identify the order of the candidates according to their positions. Bartels (1986) finds wide variation in voter perceptions of candidate positions on several salient issue dimensions in the 1980 US Presidential elections from the NES data. Lenz (2012, table 5.1, page 117-118) presents a survey where he studies several salient issues in US and European national elections (social security in 2000 US elections, EU integration in the British 1997 elections, public works jobs in the 1976 US elections, defense spending in the 1980 US elections, ideology in the 1992 US elections and Chernobyl in the 1986 Dutch elections) and shows that, in each case, less than half of the respondents could start out identifying the order of candidates correctly. These facts suggest that even in electoral races between candidates, there may be uncertainty in voters' minds about the order of candidates.

In case of electoral competitions, our results provide a new explanation for the phenomenon of incumbency advantage. It is well documented that incumbents enjoy a strong and growing advantage in US electorates - both in legislative and executive offices (Ansolabehere, Snyder and Stewart 2000, Ansolabehere and Snyder 2002). We hold that if there is incomplete information regarding wether the challenger lies to the left or right of the incumbent, then incumbency advantage may arise due to a co-ordination failure among voters. While the existing set of explanations of incumbency advantage relating to political structure (e.g. decline of the party (Cover 1977), campaign contribution and interest group activities (Jacobson 1980)) apply to legislative offices, our explanation applies to executive offices as well. In fact, our theory is particularly suited to lower offices where information regarding the challenger is harder to come by and party identification plays a smaller role. Another literature (Erikson 1995, Ansolabehere, Snowberg and Snyder 2006) suggests that the advantage of the incumbent arises from being able to corner a larger share of television time both in terms of news coverage as well as campaign advertisements. Our explanation is broadly in line with this position: the incumbent advantage stems from the voters being more informed about the incumbent than about the challenger.

1.3 Related literature

Fernandez and Rodrik (1991) was the first to show that welfare improving trade reforms may be blocked due to distributional uncertainty. Their idea is the following: if the reform passes in the minority preferred state, that information will be revealed following implementation and then it will be voted down by the majority in another round of election. On the other hand, if the status quo wins when the reform is actually majority preferred, the state is never revealed and the population retains the status quo. Thus, unlike our mechanism, their theory of status quo bias crucially hinges on there being multiple rounds of elections and partial revelation of the state in between.

Ali, Mihm and Siga (2018) provide general conditions on the nature of preference variation that leads to the ex-post welfare-maximizing alternative to be voted down in some equilibria. However, their mechanism hinges on private signals being very noisy while ours holds for arbitrarily precise individual information.

It is important to mention the formal relationship between conditions on information aggregation in the spatial model and those in the more general setting in Bhattacharya (2013). According to the Strong Preference Monotonicity (SPM) condition in Bhattacharya (2013), if the distribution of preferences is such that a randomly chosen voter is more likely to prefer P over Q for *each* prior belief over states, then information is aggregated in all equilibria. Conversely, if SPM is not satisfied, then there exist signal precisions for which a "wrong" outcome obtains in at least one equilibrium. Bhattacharya (2013) also identifies a joint condition on signal precision and preference distribution called Weak Preference Monotonicity (WPM) that has the same flavor.[6] In the spatial model, if the alternatives are ordered, both SPM and WPM are satisfied. Hence, it follows directly that information is aggregated efficiently in every equilibrium. On the other hand, when the alternatives are unordered, both SPM and WPM are violated. It is important to point out that the non-aggregating equilibrium identified by Bhattacharya (2013) is the activist voting equilibrium in the spatial model. The spatial structure allows us to unearth other equilibria: both efficient and inefficient ones in the same environment. Moreover, the current paper derives the equilibrium strategies which allows us to track the behavior of responsive (i.e., "swing") voters and provide conditions on responsive sets for the election to achieve the correct outcomes in equilibrium. These conditions throw light on the reasons for why information may or may not be aggregated in certain equilibria.

There is a parallel literature on aggregation failure in common value elections. Mandler (2012) shows that there exist non-aggregating equilibria if there is uncertainty over precision of signals. Our paper shares with Mandler's the idea that voting equilibria are sensitive to local properties of vote share functions (while full information outcomes are not). Aggregation failure due to multiple equilibria can also occur when the number of eligible voters varies across states. While Myerson (1998) shows that there always exists an information aggregating equilibrium, Ekmekci and Lauermann (2019) solve the set of equilibria and show that there exist additional non-aggregating equilibria with state-dependent electorate size. In the common value auctions literature, Atakan and Ekmekci (2014) have a similar insight where they show that information aggregation may fail in equilibrium if bidders' expected valuation is non-monotonic in their belief over states.

Persico (2004) and Martinelli (2006) show that in large committees votes may be uninformative if information is costly. We demonstrate that uninformative voting may arise (in the block voting equilibrium) due to preference diversity even if signals are free.

[6] WPM is said to be satisfied if a change in the signal makes a randomly chosen voter more switch her vote for P to Q for *each* belief over states. For a given distribution of preference, SPM holds if and only if WPM holds for every possible distribution over signals.

In all the above papers the state space is binary. Barelli et al (2020) shows that even with common values, with general state and signal spaces there may not exist *any* feasible strategy profile that aggregates information, and generically so if the state space is infinite. While aggregation failure in the papers cited earlier is due to complexity in preferences, the failure in Barelli et al (2020) is due to complexity in the information structure. Kosterina (2020) also looks at common value elections with ordered states and provides sufficient conditions on the information structure for the existence of threshold equilibria that fail to aggregate information.

The paper is organized as follows. Section 2 discusses the basic model. Section 3 discusses optimal voting behavior as a function of prior beliefs over states. Section 4 characterizes the set of equilibrium outcomes and section 5 identifies information aggregation properties in large elections for different environments. Section 6 provides a discussion of the robustness of the results. Most proofs are relegated to the appendix

2 Model

2.1 Set-up

There is an electorate composed of $n+1$ voters who vote over two policies P and Q. Alternative P wins if the proportion of votes cast in favour of P is equal to or more than $\theta \in (0,1)$, otherwise Q wins. Policy alternatives are located in the unit interval $[0,1]$. Location of each alternative is uncertain, and varies with the state of the world $\omega \in \{L, R\}$. States are assumed to be equally likely, although that plays no role in the analysis. The location of policy $A \in \{P, Q\}$ in state ω is given by the parameter $x_\omega^A \in (0,1)$. We assume that $x_L^A \leq x_R^A$ for $A \in \{P, Q\}$, i.e., in state L, both policies shift left.

We rule out two trivial cases. First, we assume that $x_\omega^P \neq x_\omega^Q$ for $\omega \in \{L, R\}$, i.e., in each state there is a choice to be made by voters between P and Q. Second, we allow at most one alternative to have the same location in both states (otherwise, there will be no uncertainty).

There are four possible orderings of the policy-state locations: (1) $x_L^P \leq x_R^P < x_L^Q \leq x_R^Q$, (2) $x_L^Q \leq x_R^Q < x_L^P \leq x_R^P$, (3) $x_L^P < x_L^Q \leq x_R^Q < x_R^P$, and (4) $x_L^Q < x_L^P \leq x_R^P < x_R^Q$, with the added restriction that only one of the two inequalities in (1) and (2) can be weak. Notice now that in (1), P is located to the left of Q in both states and in (2) P is located to the left of Q in both states. In other words, the alternatives are locationally ordered: even though the exact location is uncertain, it is known that P lies on one side of Q. On the other hand, in (3), P is located to the left of Q in state L and to the right of Q in state R. Alternatively, despite locational uncertainty, Q is known to be the central alternative and is flanked by the two possible locations of P on either side. Environment (4) is similar to

(3), with the roles of the alternatives reversed. With no loss of generality, we can restrict our attention to the environments (1) and (3).

Definition 1 *Denote case with* $x_L^P \leq x_R^P < x_L^Q \leq x_R^Q$ *(with at most one equality allowed) as the ordered alternatives environment. In this case,* P *lies to the left of* Q *in both states. Denote the case with* $x_L^P < x_L^Q \leq x_R^Q < x_R^P$ *as the unordered alternatives environment. Here,* P *lies to the left of* Q *in state* L *and to the right of* Q *in state* R.

Observe that this formulation allows the special case where a reform with uncertain consequence is competing against a known status quo. This would be the case where $x_L^Q = x_R^Q = x^Q$.

In either environment, each voter receives a noisy private signal $s \in \{l, r\}$ drawn independently from the following distribution conditional on the state

$$\Pr(l|\omega = L) = \Pr(r|\omega = R) = q \in \left(\frac{1}{2}, 1\right)$$

We will refer to q as the signal precision.

Voters have single peaked preference symmetric about the peak. Every individual has a privately known bliss point x that is drawn independently from a non-atomic distribution $F(\cdot)$ with support $[0, 1]$. We assume there is a continuous density function $f(\cdot)$ for which there is some $\alpha > 1$ such that $\frac{1}{\alpha} < f(x) < \alpha$ for all $x \in [0, 1]$. Both the realized ideal point x and realized signal s are private information to the particular voter. Henceforth, for convenience of description, by the term "type" we shall refer only to the ideal point x. The utility for a voter with type $x \in [0, 1]$ from an alternative located $y \in [0, 1]$ is given by the continuous loss function $-v(z)$, where $z = |x - y|$ and $v(0) = 0$ and for all $z > 0$, $v > 0$ and $v' > 0$.

We define a voting environment by the tuple $(F, q, \{x_\omega^A\})$. A voting game is a voting environment coupled with an electorate size n and a voting rule θ.

In terms of voter preference, what matters is the utility difference $V(x, \omega)$ between the two alternatives P and Q in state ω for an agent with ideal point x, where

$$V(x, \omega) \equiv v(|x - x_\omega^Q|) - v(|x - x_\omega^P|)$$

Denote, for $\omega \in (L, R)$, $x_\omega = \frac{x_\omega^P + x_\omega^Q}{2}$. This is the threshold indifferent type that splits the type space $[0, 1]$ into the types who prefer P and those who prefer Q. By assumption, $x_L < x_R$.

The classification of environments with those with ordered alternatives and those with unordered alternatives is crucial. In the ordered alternatives environment, $x_\omega^P < x_\omega^Q$, or $V(x, \omega)$ is decreasing in x in both states. In this scenario, the types $x < x_L$ prefer P to Q in each state (P-partisans) and the types $x > x_R$ prefer Q to P in each state (Q-partisans). Types $x \in (x_L, x_R)$ are independents: they prefer Q in state L and P in state R.

In the unordered alternatives environment, $x_L^P < x_L^Q$ and $x_R^P > x_R^Q$, or $V(\cdot,\omega)$ is decreasing in state L and increasing in state R. In this scenario, an interval of moderate types $x \in (x_L, x_R)$ are Q-partisans. There are two groups of independents: The types $x < x_L$ prefer P to Q in state L and Q to P in state R and will be referred to as the L-group, while those with $x > x_L$ have exactly opposite preference in each state and will be referred to as the R-group.

For the unordered alternatives environment only, we make two more assumptions. These assumptions are not necessary for the results on ordered alternatives environments.

A1 The loss function $v(z)$ is convex and v' is log-concave. Formally, $v'' < 0$ and $\frac{v''}{v'}$ is decreasing for all $z > 0$.

A2 Signal precision q is greater than $\widehat{q} \equiv \left(1 + \sqrt{\frac{v(x_L)v(1-x_R)}{v(x_R)v(1-x_L)}}\right)^{-1}$

Assumption A1 says that the loss increases at a faster rate as the policy moves further away, but the rate of increase is not "too fast". This assumption delivers an important monotonicity property for the function $-\frac{V(x,R)}{V(x,L)}$, making cut-off strategies optimal for the unordered alternatives environment.

Assumption A2 places a lower bound on signal precision. This is necessary to avoid trivial equilibria where everyone votes for Q. It is important to note that this assumption stacks the deck against negative results.

The loss function $v(z) = z^\alpha, \alpha > 1$ satisfies A1. In particular, if we have quadratic loss functions ($\alpha = 2$), then V is linear in x, i.e., $V(x,\omega) = (x - x_\omega)b_\omega$, where $b_\omega = 2(x_\omega^P - x_\omega^Q)$.

Figure 1 shows the voter preferences separately for ordered and unordered alternatives environments, assuming a linear V.

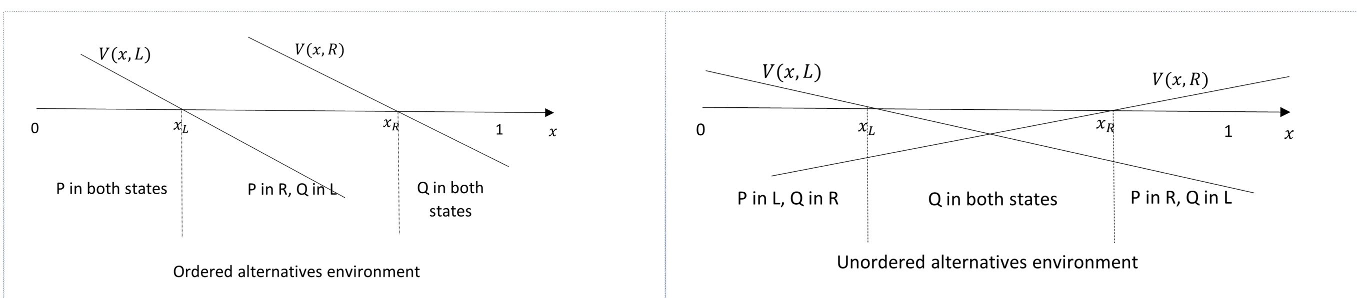


Figure 1

2.2 Strategies and equilibrium - Definition

A strategy in the game $(F, q, \{x_\omega^A\}, n, \theta)$ is a probability $\pi(x, s)$ of voting for P, for each $x \in [0, 1]$ and $s \in \{l, r\}$. Since F is nonatomic, we can consider only pure strategies $\pi(x, s) \in \{0, 1\}$. We consider Bayesian Nash equilibria with two additional restrictions on equilibrium strategy profiles: (i) type-symmetry, i.e., agents with the same private information (x, s) use the same strategy, and (ii) responsiveness, i.e., each action $\{P, Q\}$ should be played with positive ex-ante probability in equilibrium.

To obtain a formal definition of responsiveness, define by z_s the probability that a voter with signal s votes for P. We have

$$z_s = \int_0^1 \pi(x, s) f(x) dx \tag{1}$$

A pure strategy π is said to be *responsive* unless $z_l = z_s = 1$ or $z_l = z_s = 0$.

Recall that a voter conditions her strategy on the event of being decisive. Suppose that, conditioning on being pivotal, a voter holds the belief that $\Pr(\omega = R|piv) = \beta$. Denote the posterior belief obtained from β given signal $s \in \{l, r\}$ by β_s.[7]

We say that a strategy π is optimal with respect to prior belief β if for each $x \in [0, 1]$ and $s \in \{l, r\}$, she votes for P if and only if[8]

$$EV(x|\beta_s) = V(x, R)\beta_s + V(x, L)(1 - \beta_s) \geq 0, \tag{2}$$

and we denote this optimal strategy by $\pi[\beta]$.

A symmetric Bayesian Nash equilibrium is defined as a strategy profile $\pi^e(\cdot, \cdot)$ with the property that it induces a such a belief β^e conditioning on pivotality that the strategy optimal with respect to the belief β^e is the same strategy π^e. More succintly,

$$\pi^e = \pi[\beta^e] \text{ and } \beta^e = \Pr(\omega = R|piv, \pi^e) \tag{3}$$

where $\Pr(\omega = R|piv, \pi^e)$ is the belief induced by the pivotal event given that all other voters are using strategy π^e.

Our equilibrium condition is therefore a fixed point in the space of beliefs, given by any solution β^e to the equation

$$\beta = \Pr(\omega = R|piv, \pi[\beta]) \tag{4}$$

The set of fixed points of $\beta \to \Pr(\omega = R|piv, \pi[\beta])$ correspond to the entire set of responsive type-symmetric BNE provided the following properties hold:

Property R For any $\beta \in [0, 1]$, $\pi[\beta]$ is responsive.

Property I The function $\beta \to \pi[\beta]$ is injective.

[7] Thus, $\beta_r = \frac{q\beta}{q\beta + (1-q)(1-\beta)}$ and $\beta_l = \frac{(1-q)\beta}{(1-q)\beta + q(1-\beta)}$.

[8] We assume that a voter votes for P if she is indifferent, but this is innocuous since F is non-atomic.

Property R ensures that the set of responsive equilibria is equivalent to the set of BNE with positive pivot probability. By property I, in order to identify the equilibrium stratey π^e, it is enough to determine the associated pivotal belief β^e.

In order to verify these properties, we have to derive the function $\beta \rightarrow \pi[\beta]$ which we do for each environment separately in section 3.[9]

Equilibrium is described by the belief β^e such that, if all other voters used the strategy $\pi[\beta^e]$, then the belief over states conditioning on a tie would be β^e. The belief β^e acts like a prior commonly shared by all voters, but it is induced by the equilibrium strategy profile. Hence we call it the *induced prior belief.*

Austen-Smith and Banks (1996) defines sincere voting as behavior where voting strategy is optimal with respect to an exogenously given prior. The difference in case of rational voting is that the voting strategy is optimal with respect to the induced prior which is endogenously determined. We proceed with our analysis in two steps. First, in section 3 we consider voting behavior *as if* voters were voting sincerely as a function of a generic exogenous prior β. In the second step (section 4), we consider which of these values of β can arise as induced prior beliefs.

3 Voting Behavior

Suppose voters have a generic prior belief $\Pr(\omega = R) = \beta$, and the updated posterior beliefs are β_s for $s \in \{l, r\}$. Denote by $z_s(\beta)$ the associated probability of voting for P conditional on signal s, according to (1).

Denote by $t(\omega, \beta)$ the likelihood of a randomly chosen voter voting for P in state ω. This is also the expected vote share for P in state ω.

$$t(L, \beta) = q z_l(\beta) + (1 - q)\, z_r(\beta) \quad (5)$$
$$t(R, \beta) = q z_r(\beta) + (1 - q)\, z_l(\beta) \quad (6)$$

In this section, our main objective is to "describe" the function $t(\cdot, \cdot)$ in an environment $(F, q, \{x_\omega^A\})$.

At this stage, we introduce an important definition. We call a voter with ideal point x a *responsive type* if $\pi(x, l) \neq \pi(x, r)$. For such a voter, the voting behavior depends on the signal she obtains. A responsive set is the set of responsive types in a given strategy profile.

We now separately describe the optimal strategy, responsive sets and vote share function for environments with ordered and unordered alternatives.

[9] This approach to equilibrium in terms of beliefs was developed in Bhattacharya (2013), where, fact 1 held due to the presence of partisans and fact 2 was implicitly true.

3.1 Ordered alternatives

Recall that, in the unordered alternatives case, types $x < x_L$ are P-partisans, types $x > x_R$ are Q-partisans and types in $[x_L, x_R]$ are independents who prefer P only in state R.

A voter with type x, signal s and belief β, votes for P if only if $EV(x|\beta_s) \geq 0$. Since V is decreasing in x in both states, the optimal strategy is to vote for P iff x is to the left of a cut-off $x_s(\beta)$, given implicitly by

$$\frac{V(x,R)}{-V(x,L)} = \frac{1-\beta_s}{\beta_s} \tag{7}$$

Observe that the function $-\frac{V(x,R)}{V(x,R)}$ is positive only in $[x_L, x_R]$, so the cut-off must lie in this interval. For the independent types, the term $\frac{V(x,R)}{-V(x,L)}$ is the ratio of benefit-to-cost from P winning. Alternatively, this is the likelihood ratio over states (L to R) at which type x is indifferent between voting P and voting Q.[10] This ratio decreases in x, implying that among the independents, those closer to x_L are more prone to voting for P.

Figure 2 depicts the function $-\frac{V(x,R)}{V(x,L)}$ for the ordered alternatives, for a linear V.

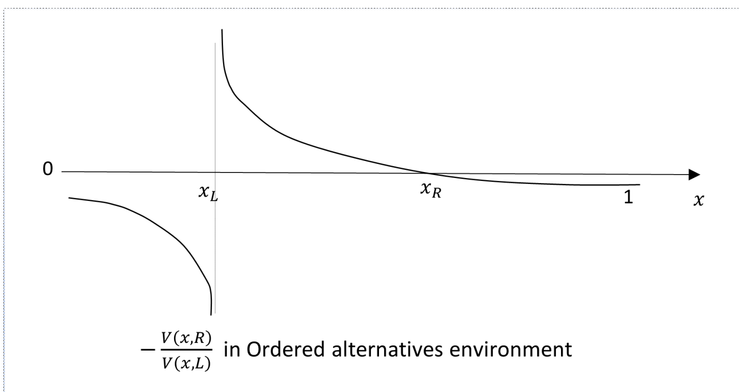


Figure 2

Formally, the following Lemma describes $\pi[\beta]$.

Lemma 1 *Consider an environment with ordered alternatives. Given a prior belief $\beta \in [0,1]$ and signal $s \in \{l, r\}$ there exists a cut-off $x_s(\beta) \in [x_L, x_R]$ such that the optimal strategy is to vote for P if $x \leq x_s(\beta)$ and Q if $x > x_s(\beta)$. Moreover, $x_s(\cdot)$ is an increasing function with (i) $x_r(\beta) > x_l(\beta)$ for $\beta \in (0,1)$, (ii) $x_r(0) = x_l(0) = x_L$, and (iii) $x_r(1) = x_l(1) = x_R$.*

[10] Alternatively, $\frac{1}{1-\frac{V(x,R)}{V(x,L)}}$ is the "threshold of doubt" where a type x switches her decision.

Proof. In Appendix ■

In the optimal strategy, there is a responsive set $[x_l, x_r]$ of types that vote for P if they get signal r and Q if they get signal l. The types to the left (right) of the responsive set always vote for P (Q) respectively. Such voting behavior mimics the desctiption of moderate swing voters voting informatively while the more extreme types vote uninformatively for their ex-ante preferred alternative.

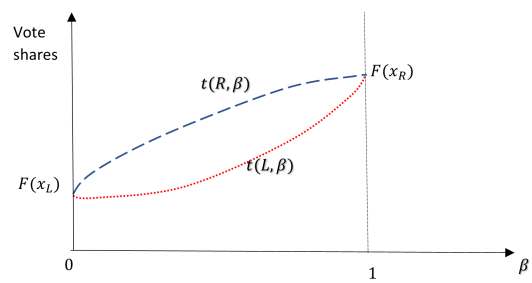
For the ordered alternatives environment, Property I is verified since $x_s(\beta)$ is strictly increasing and Property R is verified since $x_s(\beta) \in [x_L, x_R]$.

From Lemma 1, we have that $z_s(\beta) = F(x_s(\beta))$. Using this in expressions (5) and (6), we obtain the vote share functions for the ordered alternative case, which is described in the following Proposition.

Proposition 1 *Consider an environment with ordered alternatives. For each state ω,the expected vote share $t(\omega, \beta)$ for P strictly increases with β. Moreover, (i) $t(L, \beta) < t(R, \beta)$ for all $\beta \in (0, 1)$, (ii) $t(L, 0) = t(R, 0) = F(x_L)$, and (iii) $t(L, 1) = t(R, 1) = F(x_R)$.*

Proposition 1 states that as the prior belief places progressively higher weight on the state more favourable to P, the expected share of votes for P increases in both states.

The expected vote shares in the two states are plotted against the prior in Figure 3



Expected Vote share functions with ordered alternatives

Figure 3

3.2 Unordered alternatives

Recall that in this environment, types $x < x_L$ (L-group) prefer P only in state L while types $x > x_R$ (R-group) prefer P only in state R. Types in (x_L, x_R) prefer Q in both states.

With unordered alternatives, $EV(x|\beta_s) \geq 0$ if and only if

$$-\frac{V(x,R)}{V(x,L)} \leq \frac{1-\beta_s}{\beta_s} \text{ and } x \leq x_L, \text{ or} \tag{8}$$

$$-\frac{V(x,R)}{V(x,L)} \geq \frac{1-\beta_s}{\beta_s} \text{ and } x \geq x_R \tag{9}$$

Assumption A1 made on curvature of the loss function now guarantees the added monotonicity on the $-\frac{V(x,R)}{V(x,L)}$ function which is necessary for optimal strategies to have a cut-off structure.

Lemma 2 *Assume A1. In the unordered alternatives environment,* $-\frac{V(x,R)}{V(x,L)}$ *is increasing in* $[0, x_L)$ *and in* $(x_L, 1]$.

Proof. In appendix[11] ■

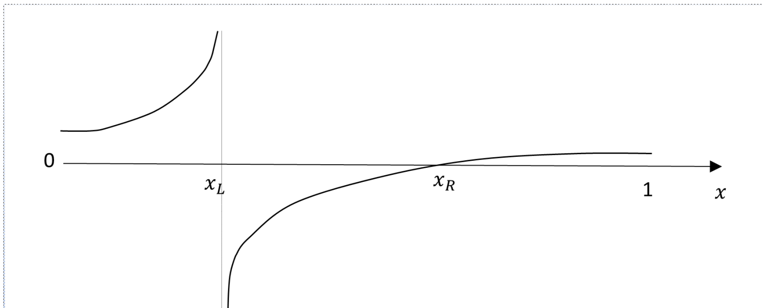
For the R-group, $-\frac{V(x,R)}{V(x,L)}$ is the benefit-to-cost ratio from P winning. According to the Lemma, this is higher for types closer to 1. Similarly, for the L-group, the net benefit-to-cost ratio from P winning is $-\frac{V(x,L)}{V(x,R)}$ which is decreasing in x and thus higher for types closer to 0. The main implication for voting behavior is that if an independent type x votes for P for a certain belief, the more extreme types in the same group also vote for P.

Lemma 3 *Assume A1. In the unordered alternatives environment,* $-\frac{V(0,R)}{V(0,L)} > -\frac{V(1,R)}{V(1,L)}$

Proof. In appendix[12] ■

Lemma 3 ensures that there is no belief for which there are two types - one in L-group and one in R-group, both voting for P. In other words, if some type in the L-group votes for P, all types in the R-group vote for P, and vice versa.

The function $-\frac{V(x,R)}{V(x,L)}$ for the unordered alternatives environment is depicted for a linear V in Figure 4.



$-\frac{V(x,R)}{V(x,L)}$ in Unordered alternatives environment

[11] An important step in the proof of this Lemma follows from Lemma 2.2 in Anderson et al (1993).

[12] This result follows from convexity of v alone. I thank Rahul Mukherjee for providing a proof of this Lemma.

Figure 4

The following Lemma formally describes the optimal strategy function $\pi[\beta]$ in the unordered alternatives environment.

Lemma 4 *Consider an environment with unordered alternatives and assume A1. For each signal $s \in \{l, r\}$, there exist thresholds $\underline{\beta^s}$ and $\overline{\beta^s}$ with the feature that:*

(a) for $\beta \in [0, \underline{\beta^s}]$, there is a cut-off $x_s(\beta) \in [0, x_L]$ decreasing in β such that the optimal strategy is to vote for P if $x \leq x_s(\beta)$ and for Q if $x > x_s(\beta)$;

(b) for $\beta \in (\underline{\beta^s}, \overline{\beta^s})$ it is optimal for all $x \in [0, 1]$ to vote for Q; and

(c) for $\beta \in [\overline{\beta^s}, 1]$, there is a cut-off $x_s(\beta) \in [x_R, 1]$ decreasing in β such that the optimal strategy is to vote for P if $x \geq x_s(\beta)$ and for Q if $x < x_s(\beta)$.

Moreover, $x_r(\beta) < x_l(\beta)$ whenever $\beta \in (0, \underline{\beta^r}) \cup (\overline{\beta^l}, 1)$; $x_r(0) = x_l(0) = x_L$, and $x_r(1) = x_l(1) = x_R$.

Proof. In appendix. ■

From the above Lemma, we have for unordered alternatives environment

$$z_s(\beta) = \begin{cases} F(x_s(\beta)) \ if \ x_s(\beta) \leq x_L \\ 1 - F(x_s(\beta)) \ if \ x_s(\beta) \geq x_R \end{cases} \tag{10}$$

The function $z_s(\beta)$ is continuous, falling in $[0, \underline{\beta^s}]$, zero in $(\underline{\beta^s}, \overline{\beta^s})$ and increasing in $[\overline{\beta^s}, 1]$. If the ranges $(\underline{\beta^l}, \overline{\beta^l})$ and $(\underline{\beta^r}, \overline{\beta^r})$ overlap, then we will have an nonempty interval of priors β for which all types vote for Q irrespective of the signal. The informativeness assumption A2 ensures that such is not the case, and therefore the pivot probability is positive for all β.[13]

Claim 1 *Assume A1 and A2. If $q > \widehat{q}$, we have $\overline{\beta^r} < \underline{\beta^l}$*

Proof. In appendix ■

Under A1 and A2, both property R and I are verified in the unordered alternatives environment. R is true since $z_s(\beta) \in (0, 1)$ for at least one s for all β. For property I, notice that $x_s(\beta)$ is locally strictly decreasing for at least one s for all β.

The structure of responsive sets in the unordered alternatives environment is described in the following corollary to Lemma 4.

Corollary 1 *Under assumptions A1 and A2, the responsive sets in the unordered alternatives environment are as follows*

(a) When $\beta \in \left(0, \underline{\beta^r}\right)$, the responsive set (x_r, x_l) is entirely in the L-group

[13] Notice that Lemma 3 implies that $\widehat{q} > \frac{1}{2}$ and A2 is indeed a restriction on the signal precision.

(b) When $\beta \in \left(\underline{\beta^r}, \overline{\beta^r}\right)$ *, the responsive set* $(0, x_l)$ *is entirely in the L-group*

(c) When $\beta \in \left(\overline{\beta^r}, \underline{\beta^l}\right)$ *, the responsive set is* $(0, x_l) \cup (x_r, 1)$*:* $(0, x_l)$ *lies in the L-group and* $(x_r, 1)$ *lies in the R-group*

(d) When $\beta \in \left(\underline{\beta^l}, \overline{\beta^l}\right)$ *, the responsive set* $(x_r, 1)$ *is entirely in the R-group*

(e) When $\beta \in \left(\overline{\beta^l}, 1\right)$ *, the responsive set* (x_r, x_l) *is entirely in the R-group*

Responsive types in the L-group vote P if and only if they get the l-signal and the responsive types in the R-group vote P if and only if they get the r-signal

Thus, for beliefs strongly supporting state L, the responsive set is in the L-group while for beliefs strongly supporting state R, the responsive set is in the R-group. Perhaps interestingly, for moderate beliefs the responsive set is itself split. Only the extreme types from both ends are responsive and vote for P on getting opposite signals.

The following proposition describes the expected vote share $t(\omega, \beta)$ for P as a fucntion of the prior β.

Proposition 2 *Consider an environment with unordered alternatives and assume A1 and A2. Both* $t(L, \beta)$ *and* $t(R, \beta)$ *are decreasing in* $(0, \overline{\beta^r})$ *and increasing in* $(\underline{\beta^l}, 1)$*.There exists some belief* $\beta^* \in (\overline{\beta^r}, \underline{\beta^l})$ *such that for* $\beta < \beta^*$*,* $t(L, \beta) > t(R, \beta)$*, for* $\beta > \beta^*$*,* $t(L, \beta) < t(R, \beta)$ *and for* $\beta = \beta^*$*,* $t(L, \beta) = t(R, \beta)$*. Moreover,* $t(L, 0) = t(R, 0) = F(x_L)$ *and* $t(L, 1) = t(R, 1) = 1 - F(x_R)$*.*

Proof. See Appendix. ■

Figure 5 depicts the vote share functions for the unordered alternatives environment.[14]

[14] This figure has been drawn based on the following environment: $v(z) = z^2$, $q = 0.8$, $x_L^P = 0.125$, $x_L^Q = 0.375$, $x_R^Q = 0.625$, $x_R^P = 0.875$, i.e., $x_L = 0.25$ and $x_R = 0.75$. The pdf of x is $f(x) = \begin{cases} 1 & \text{for } x \in [0, 0.25] \\ 3 & \text{for } x \in [0.75, 1] \\ 0 & \text{otherwise} \end{cases}$. This gives us $\underline{\beta^r} = \frac{1}{13}$, $\overline{\beta^r} = \frac{3}{7}$, $\underline{\beta^l} = \frac{4}{7}$ and $\overline{\beta^l} = \frac{12}{13}$. .

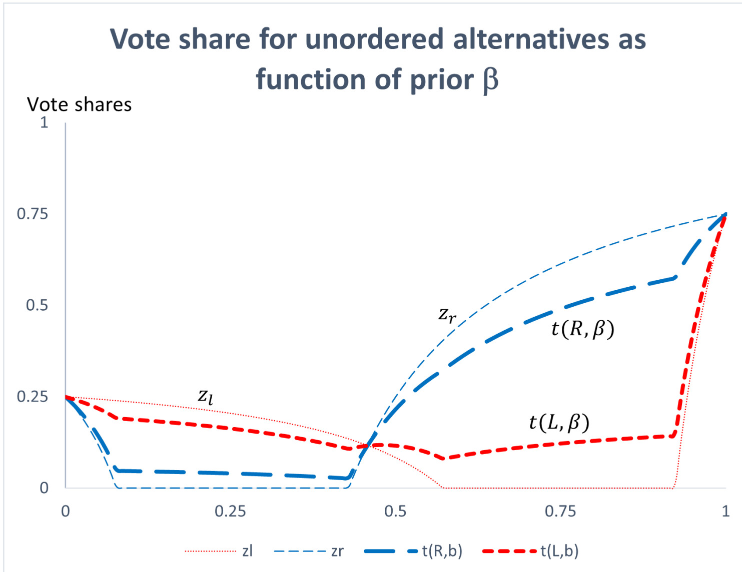


Figure 5

Before concluding this section, we make a few observations.

First, from Proposition 1 and 2, in both environments, $t(\omega, \beta)$ is continuous in β and bounded above 0 and below 1.

Second, the functions $t(L, \beta)$ and $t(R, \beta)$ are not exactly U-shaped in the unordered environment. They are initially falling and finally rising, but the shape in the intermediate interval $(\overline{\beta^r}, \underline{\beta^l})$ depends on the parameters.

Third, Bhattacharya (2013) already allows us to see why we will have contrasting results in the two types of environment. The ordered alternatives environment satisfies the Strong Preference Monotonicity condition since $t(L, \beta) < t(R, \beta)$ for all $\beta \in (0, 1)$ (by Proposition 1). Hence,. FIE holds for every information structure. On the other hand, by Proposition 2, the vote share functions "cross" at β^*. Therefore, Weak Preference Monotonicity is violated for all $q > \widehat{q}$, and there exists an equilibrium with induced prior converging to β^* where aggregation fails as P receives equal vote share in both states.

4 Equilibrium Analysis

In this section, we borrow the technique developed in Bhattacharya (2013) to solve for limit equilibria in a game. Given the expected vote share function $t(\omega, \beta)$, we identify which values of β will satisfy equation 4, i.e., yield optimal strategies $\pi[\beta]$ that will return the same belief β conditioning on pivotality.

We first show existence of equilibria for finite population and then provide a characterization of limit values of equilibrium induced priors for large electorates.

4.1 Finite population

Consider a candidate induced prior belief β. When all other voters use the strategy $\pi[\beta]$, the probability of a random voter voting for P in state ω is $t(\omega, \beta)$. Under a threshold rule θ a voter is pivotal if $\lfloor n\theta \rfloor$ votes are cast for the policy P from among the remaining n voters, where $\lfloor t \rfloor$ denotes the largest integer weakly less than t. The probability that a given voter is pivotal in state ω is given by:

$$\Pr(piv|\beta, \omega) = \binom{n-1}{\lfloor n\theta \rfloor} (t(\omega, \beta))^{\lfloor n\theta \rfloor} (1 - t(\omega, \beta))^{n-1-\lfloor n\theta \rfloor} \tag{11}$$

Since $0 < t(\omega, \beta) < 1$ for either environment, we must have $\Pr(piv|\beta, \omega) > 0$.

From the equiliibrium condition (4), we obtain,

$$\frac{\beta}{1-\beta} = \frac{\beta\,(R|piv, \beta)}{\beta\,(L|piv, \beta)} = \frac{\Pr(piv|\beta, R)}{\Pr(piv|\beta, L)},$$

which, using the pivot equations (11), gives us

$$\begin{aligned} \beta &= \frac{H(\beta, n, \theta)}{1 + H(\beta, n, \theta)} \text{ where} \\ H(\beta, n, \theta) &= \frac{(t(R, \beta))^{\lfloor n\theta \rfloor} (1 - t(R, \beta))^{n-1-\lfloor n\theta \rfloor}}{(t(L, \beta))^{\lfloor n\theta \rfloor} (1 - t(L, \beta))^{n-1-\lfloor n\theta \rfloor}} \end{aligned} \tag{12}$$

Any solution to equation (12) is an equilibrium belief denoted by β_θ^n (indexing by the voting rule and number of voters), and the equilibrium strategy profile is given by $\pi[\beta_\theta^n]$.

Equation (12) admits a solution, since that the right hand side is continuous in β and bounded above 0 and below 1, but the left hand side continuously changes from 0 to 1. It is important to note that β_θ^n must lie in $(0, 1)$ from the equation (12).

We cannot characterize β_θ^n, but we can characterize the limit values of β_θ^n, which is what we do next.

4.2 Large Electorates

Fixing an environment and voting rule θ, consider a sequence of games by letting the number of voters grow unboundedly, and denote the limit of the equilibrium sequence $\langle \beta_\theta^n \rangle_{n=1}^\infty$ by β_θ^0.[15] Since condition (12) must be satisfied along the sequence, we obtain the

[15] The existence of such a limit point (more formally, accumulation point of a subsequence) is guaranteed by the fact that the space of beliefs is compact.

following *limit equilibrium condition.*

$$\beta_\theta^0 = \lim_{n\to\infty}\left[\frac{H(\beta_\theta^n,n,\theta)}{1+H(\beta_\theta^n,n,\theta)}\right] \tag{13}$$

The vote share functions $t(\omega,\beta)$, $\omega\in\{0,1\}$ are determined entirely by the environment $(F,q,\{x_\omega^A\})$. Given $t(\omega,\beta)$, we obtain the correspondence $\Theta:[0,1]\to[0,1]$, where $\Theta(\beta)$ is the set of voting rules θ for which there exists a sequence of equilibria with induced priors β_θ^n converging to β. We then "invert" the correspondence to determine the set of limit values of induced prior beliefs for each value of θ.

For values of β such that $t(L,\beta)\neq t(R,\beta)$, define the function

$$\theta^*(\beta) = \frac{\log\frac{1-t(L,\beta)}{1-t(R,\beta)}}{\log\frac{t(R,\beta)(1-t(L,\beta))}{t(L,\beta)(1-t(R,\beta))}} \tag{14}$$

The following remark establishes a few important properties of the function $\theta^*(\beta)$.

Remark 1 *$\theta^*(\beta)$ is continuous and lies strictly between $t(L,\beta)$ and $t(R,\beta)$. Moreover, if both $t(L,\beta)$ and $t(R,\beta)$ are strictly increasing or strictly decreasing for some β, then so is $\theta^*(\beta)$.*

Proof. In appendix ■

The second part of the remark ensures that $\theta^*(\beta)$ is increasing in the ordered alternatives environment (from Proposition 1). In the unordered alternatives case, $\theta^*(\beta)$ is decreasing in the range $(0,\overline{\beta^r})$ and increasing in $(\underline{\beta}^l,1)$ (from Proposition 2). $\theta^*(\beta)$ can, however, be non-monotonic in the range $(\overline{\beta^r},\underline{\beta}^l)$.

In order to state the (partial) characterization result in a concise way, we need another definition. We say that $\beta\in[0,1]$ is *regular* if either (i) $\theta^*(\beta)$ is well-defined and strictly monotonic in a neighborhood of β, or (ii) $t(L,\beta)=t(R,\beta)\neq\theta$.[16]

Proposition 3 *For an ordered alternatives environment, define the correspondence $\Theta(\beta)$ as follows:*

$$\Theta(\beta)=\Theta^O(\beta)\equiv\begin{cases}(i)\ \text{For } \beta\in(0,1), \Theta(\beta)=\theta^*(\beta)\\ (ii)\ \Theta(0)=\{\theta:\theta\leq F(x_L)\}\\ (iii)\ \Theta(1)=\{\theta:\theta\geq F(x_R)\}\end{cases}$$

For an unordered alternatives environment, define the correspondence $\Theta(\beta)$ as follows:

$$\Theta(\beta)=\Theta^U(\beta)\equiv\begin{cases}(i)\ \text{For } \beta\in(0,\beta^*)\cup(\beta^*,1),\ \Theta(\beta)=\theta^*(\beta)\\ (ii)\ \Theta(\beta^*)=\{\theta:\theta\in(0,1)\}\\ (iii)\ \Theta(0)=\{\theta:\theta\geq F(x_L)\}\\ (iv)\ \Theta(1)=\{\theta:\theta\geq 1-F(x_R)\}\end{cases}$$

[16] The local monotonicity property can also be written as $\frac{d\theta^*}{d\beta}\neq 0$, but at $\overline{\beta^s}$ and $\underline{\beta}^s$, the right hand derivative differs from the left hand derivative. Local monotonicity holds at these points as long as both the right hand derivative and left hand derivative are positive or both negative.

For both classes of environments, given any $\beta \in [0,1]$, *there exists a sequence of equilibria with induced prior beliefs* β_θ^n *converging to* β *only if* $\theta \in \Theta(\beta)$. *Conversely, there is a sequence of equilibria with induced prior* β_θ^n *converging to* β *if* $\theta \in \Theta(\beta)$ *provided* β *is regular.*

Proof. See supplementary Online appendix ■

The above proposition characterizes the limits of all equilibrium sequences for (almost) every voting rule. It says that the equilibrium beliefs in the limit are determined by the shape of the vote share functions $t(L,\beta)$ and $t(R,\beta)$. If β produces unequal vote shares in the two states, then there is at most one voting rule $\theta^*(\beta)$ (and exactly one as long as β is regular) that supports an equilibrium sequence converging to β. If β produces equal vote shares in the two states, there exists a continuum of voting rules that support an equilibrium sequence converging to β, as described in the correspondence $\Theta(\beta)$. In particular, in the unordered alternatives environment, for *every* $\theta \in (0,1)$ there is an equilibrium with beliefs converging to β^* except possibly for the case $\theta = t(\omega, \beta^*)$.[17]

Figure 6 separately shows the correspondence $\Theta(\beta)$ for the ordered alternatives case ($\Theta^O(\beta)$) and for the unordered alternatives case ($\Theta^U(\beta)$). In both cases, the correspondence $\Theta(\beta)$ is marked with the thick black line.

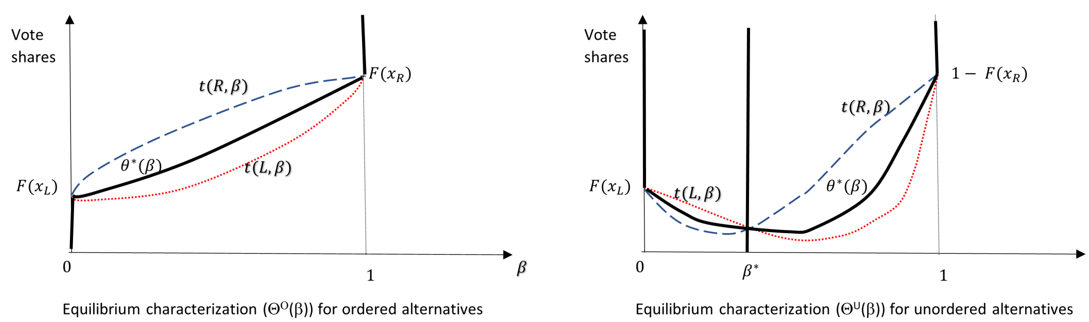


Figure 6

The formal proof of Proposition 3 follows Lemma 1, 2 and 3 in Bhattacharya (2013) with minor variations and is furnished in the appendix. We provide a brief roadmap for the proof below.

[17] The indeterminacy at $\theta = t(\omega, \beta^*)$ matters for voting rules that are trivial in the sense that they select P as the winner irrespective of the state under full information.

4.2.1 Proof sketch

First, we demonstrate that a necessary condition for having induced priors converging to any belief β is that pivot probabilities in the two states must be equal at β, given the voting rule. Suppose that, for some $\beta^0 \in (0,1)$, there exists an equilibrium sequence β^n_θ converging to β^0. If $t(L,\beta^0) = t(R,\beta^0)$, then trivially, pivot probabilities would be equal across states. So, suppose that $t(L,\beta^0) < t(R,\beta^0)$. Ignoring the integer issue, the limit equilibrium condition boils down to

$$\frac{\beta^0}{1-\beta^0} = \lim_{n\to\infty}\left[\frac{(t(R,\beta^n_\theta))^\theta\,(1-t(R,\beta^n_\theta))^{1-\theta}}{(t(L,\beta^n_\theta))^\theta\,(1-t(L,\beta^n_\theta))^{1-\theta}}\right]^n$$

Since the left hand side is a positive finite number, the expression inside square brackets on the right hand side must converge to 1.

$$\left(t(L,\beta^0)\right)^\theta\left(1-t(L,\beta^0)\right)^{1-\theta} = \left(t(R,\beta^0)\right)^\theta\left(1-t(R,\beta^0)\right)^{1-\theta} \tag{15}$$

In other words, the pivot probability must be the same in both states at β^0.

Thus, for $t(L,\beta^0) \neq t(R,\beta^0)$, the necessary condition identifies a unique voting rule θ for which there may be a sequence $\beta^n_\theta \to \beta^0$. This voting rule satisfies equation (15) and is given by $\theta^*(\beta^0)$ as in equation 14. To see why it must be the case that $t(L,\beta^0) < \theta^*(\beta^0) < t(R,\beta^0)$, notice that voting rules $\theta \leq t(L,\beta^0)$ produce strictly higher pivot probability in state L and $\theta \geq t(R,\beta^0)$ produce strictly higher pivot probability in state R.

Given that β^0 is regular, we now show that there exists an equilibrium sequence with $\beta^n_\theta \to \beta^0$ when $\theta = \theta^*(\beta^0)$. Assume wlog that θ^* is increasing around β^0. For beliefs β' in the left-neighbourhood of β^0, we must have $\theta^*(\beta') < \theta^*(\beta^0)$. For such a β', RHS of (15) is greater than the LHS. Therefore, $H(\beta', n, \theta^*(\beta_0)) \to \infty$, where the H function is obtained from (12) [18] Similarly, for beliefs β' in the left-neighbourhood of β^0, we must have $H(\beta', n, \theta^*(\beta_0)) \to 0$. By continuity, there must be a $\beta_n^{\theta^*(\beta^0)}$ converging to β that satisfies equation (12), i.e., $\frac{\beta}{1-\beta} = H(\beta, n, \theta^*(\beta_0))$.

Next, consider values of β such that $t(L,\beta) = t(R,\beta) = t$ (say). These are $\{0,1\}$ for the ordered alternatives environment and $\{0,\beta^*,1\}$ for the unordered alternatives environment. Equality of pivot probabilities across states is trivially satisfied for all voting rules $\theta \in (0,1)$ for these beliefs. However, for the corner beliefs, we can further restrict the set of candidate beliefs to either $\theta \geq t$ or $\theta \leq t$. Whether $\theta \leq t$ or $\theta \geq t$ will obtain for a particular environment depends on two factors: For a sequence $\beta^n \to \beta \in \{0,1\}$, whether $t(\omega,\beta^n)$ approaches t from above or below, and (ii) the sign of $t(L,\beta^n) - t(R,\beta^n)$ close to β.

For a particular illustration, consider $\beta = 0$ for the unordered alternatives case. Here, $t = F(x_L)$. If we are to have any equilibrium sequence $\beta^n \to 0$, it must be the case that the

[18] This utilizes the following observation: Suppose $g(x,y,\theta) = \frac{x^\theta(1-x)^{1-\theta}}{y^\theta(1-y)^{1-\theta}}$ for some $1 > x > y > 0$. Then, we must have $\frac{\partial g(x,y,\theta)}{\partial\theta} > 0$.

probability of P getting θ share of votes is stictly higher in state L than in state R along the sequence. Since $t(R, \beta^n) < t(L, \beta^n) < t$ in the neighbourhood of 0, this is satisfied if $\theta \geq t$. If on the other hand, $\theta < t$, then we will have $\theta < t(R, \beta^n) < t(L, \beta^n)$ which will lead to a higher pivot probability in state R, which is a contradiction. Therefore, $\theta \geq F(x_L)$ is consistent with limit value of induced prior being 0 while $\theta < F(x_L)$ is not. Finally, we use a similar construction as the case with unequal vote shares to show the existence of a sequence of equilibria with induced prior converging to 0, for any $\theta \geq F(x_L)$.

Finally, consider the case $\beta = \beta^*$, and again write $t(L, \beta^*) = t(R, \beta^*) \equiv t$. Unlike the corner cases, it is possible for a sequence β^n to approach β^* either from the left or right. We use the same logic to show that for $\theta > t$, there is an equilibrium sequence approaching β^* from one direction and for $\theta < t$, there is a corresponding sequence approaching β^* from the other direction.

5 Election Outcomes and Information Aggregation

In this section, we ask whether election outcomes under incomplete information converge to the complete information outcome. In our set-up, the state can be identified if signals were pooled. Thus, if the electoral outcome approaches the complete information outcome, we say that the election *aggregates* all the dispersed private information. Our main result is that information aggregation is guaranteed when the alternatives are ordered but not when alternatives are unordered.

In this section, we concentrate on *consequential* voting rules, i.e., rules under which different alternatives would be elected in the two states, in the full information benchmark. Under ordered alternatives, we assume that $F(x_L) < \theta < F(x_R)$, i.e., a θ-majority of the population prefers P in state R and Q in state L. In case of unordered alternatives, assume wlog that $F(x_L) < 1 - F(x_R)$. Now, the consequential rules are defined by $F(x_L) < \theta < 1 - F(x_R)$: these are the rules for which a θ-majority prefers P in state R and Q in state L.[19]

Given a consequential rule and the environment under consideration, we say that an equilibrium sequence is *Full Information Equivalent* if under that sequence, the probability that P wins in state R and Q wins in state L converges to 1.[20]

The next proposition establishes that information aggregation is guaranteed when the alternatives are ordered.

[19] Our framework allows us to discuss the information aggregation property for non-consequential rules too. We do so in a later section.

[20] We have chosen our parameters in such a way that the full information outcome is the same for both ordered and unordered environments, purely for brevity of exposition.

Proposition 4 *Consider an environment with ordered alternatives and a consequential voting rule. All equilibrium sequences are full information equivalent and the induced prior belief in all equilibrium sequences converge to the a unique limit.*

The proof follows from remark 1 and Proposition 3. The idea can be readily seen from figure 7. There is a unique value β_a for which $\theta^*(\cdot) = \theta$. Since $t(L, \beta_a) < \theta < t(R, \beta_a)$, by the strong Law of Large numbers, P wins almost surely in state R and Q in state L.

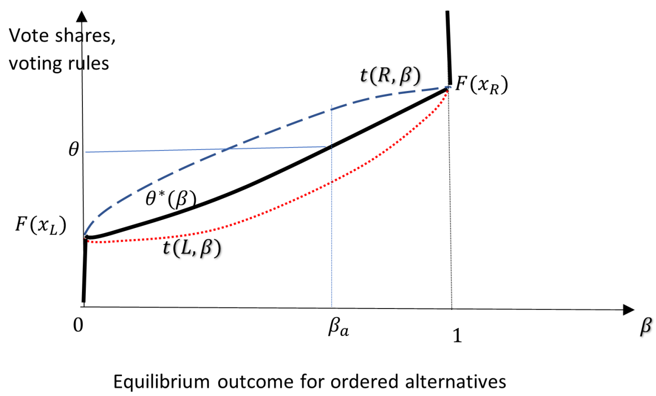


Figure 7

Now we turn to the behavior of voters in this equilibrium. Denote the type x with $F(x) = \theta$ as the θ-median. There are two properties of the responsive set that drives information aggregation. First, since θ-median is contained in the responsive set and the types on two sides of the set vote for opposite alternatives, the responsive voters are *influential*: they can affect the outcome of the election. Second, the preferences of the responsive voters are *aligned* with the complete information outcome: they prefer P in state R and Q in state L. Thus, by voting for P on receiving signal r and for Q on receiving signal l, they "swing" the election in favor of the ex-post θ-majority preferred alternative.

It is worthmentioning that the above proposition is stronger than Theorem 1 in Bhattacharya (2013) in that it establishes uniqueness of the limit of induced prior.

The next proposition describes the outcomes in the unordered alternatives case.

Proposition 5 *Consider an environment with unordered alternatives and a consequential voting rule. Assume A1 and A2. Each equilibrium sequence satisfies exactly one of the following limit properties: (i) The induced prior converges to* 0 *and Q wins almost surely in each state, (ii) the induced prior converges to β^* and the Q wins almost surely in each* *state; and (iii) the induced prior converges to some value in* $(\beta^*, 1)$ *and the outcome is full information equivalent.*

Figure 8 illustrates the three different kinds of equilibrium sequences in this environment. The three possible limit values of the induced prior are β_1, $\beta_2 (= \beta^*)$ and $\beta_3 (= 0)$. Information is aggregated in the sequence with beliefs converging to β_1 since $t(L, \beta_1) < \theta < t(R, \beta_1)$. In the two other sequences, we have $t(L, \beta) = t(R, \beta) < \theta$, and Q wins in both states.

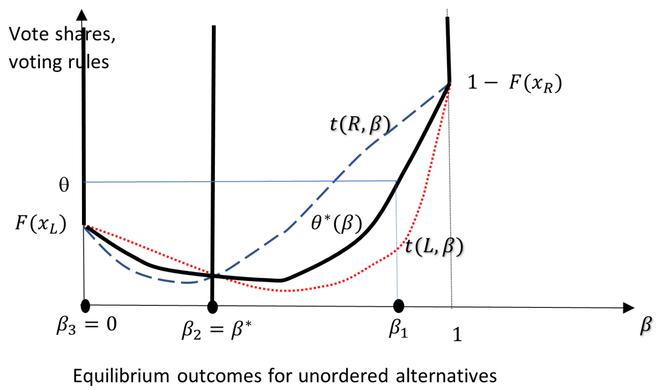


Figure 8

Observe that, with a general distribution of voter preferences, we cannot rule out the possibility of multiple information-aggregating sequences with different limit beliefs. Multiple solutions to $\theta^*(\beta) = \theta$ can arise in $(\beta^*, \underline{\beta}^l)$ since $\theta^*(\cdot)$ can be non-monotonic in this interval. We can avoid θ being in this range by making $F(x_L)$ large enough, since we must have $\theta > F(x_L)$ by definition.

The following remark shows that if the L-group and R-group are not too dissimilar in size, then there are exactly three possible limit values of the induced prior belief.

Remark 2 *Consider an environment with unordered alternatives and a consequential voting rule. Assume A1 and A2. If* $F(x_L) \geq q\left(1 - F(x_r\left((\underline{\beta}^l\right)\right)$, *then the induced prior belief in each equilibrium sequence converges to one of three possible values* 0, β^* *and some* $\beta_1 \in (\underline{\beta}^l, 1)$. *In sequences with beliefs converging to* 0 *and* β^*, Q *wins almost surely in each state. Sequences with beliefs converging to* β_1 *are full information equivalent.*

5.1 Equilibrium behavior

We now study the voting behavior in the different equilibria in the unordered case.

5.1.1 Information Aggregating Equilibrium

Existence of an equilibrium sequence with induced prior converging to some value in $(\beta^*, 1)$ follows from continuity of $\theta^*(\cdot)$ and the fact that $\theta^*(\beta^*) < \theta < \theta^*(1)$. This sequence satisfies FIE, since $t(L, \beta) < t(R, \beta)$ in $(\beta^*, 1)$.

In this sequence, the limit belief β_a places large enough weight on state R that the responsive set lies in the R-group. Thus, the alignment condition for responsive voters is satisfied. Since the θ-median from the right (the type x satisfying $\theta = 1 - F(x)$) is contained in the responsive set, the responsive voters are influential in affecting the voting outcome, and we obtain the complete information outcome in each state.

5.1.2 Block Voting Equilibrium

Next, consider the equilibrium sequence with induced prior converging to 0. In this sequence, the responsive set lies in the L-group. But such a responsive set can never contain the θ-median because $F(x_L) < \theta$. The responsive set constitutes a vanishing fraction of types close to x_L.

Observationally, limit behavior in this equilibrium resembles block voting: the R-group vote in favor of Q while the L-group votes for P irrespective of the private signals received. Such uninformative voting behavior is fuelled by the belief that (almost) everyone else is also going to vote uninformatively.

As an aside, notice that while there is an uninformative equilibrium with beliefs converging to 0, there is no equilibrium sequence with beliefs converging to 1. In the former case, $t(R, \beta^n) < t(L, \beta^n) < \theta$ for large enough n, implying that a vote share of exactly θ for P is much more likely in sate 0 than in state R which is consistent with $\beta^n \to 0$. On the other hand, suppose voters believe β is close to 1. In this case, $t(R, \beta^n) > t(L, \beta^n) > \theta$ for large enough n, making the pivotal event much more likely in state L, which is inconsistent with $\beta^n \to 1$.

5.1.3 Activist Voting Equilibrium

Finally, consider the equilibrium sequence with $\beta^n \to \beta^*$. At β^*, the belief is moderate enough so that the responsive set is split between the L-group and R$-$group. An equilibrium obtains at this belief since the pivot probabilities in the two states are equal at β^*. At this belief, only the extreme types in either group vote for P if they get a signal that favors P. On the other hand, the non-extreme types in both groups vote for Q (the "central" alternative) at this moderate belief, and Q wins in both states.

Observationally, the ones with largest magnitude of utility difference (the "activists") in the two groups appear to be voting for the same alternative for opposite reasons. Such behavior is quite different from the conventional description that the moderate voters are the ones that swing the election one way or the other depending on the information they receive.

Notice that in both the block voting equilibrium and the activist voting equilibrium, the responsive voters fail to be influential, and therefore the same alternative wins in both states. The alignment condition also fails in both these cases. Thus, both the necessary conditions for FIE with consequential rules fail in these equilibria.[21]

6 Discussion

In the main body of the paper, we have shown that information may not be aggregated due to a co-ordination failure among voters and provided a description of the possible types of equilibrium behavior. We conclude by providing some comments on robustness of results and multiplicity of equilibria.

6.1 Other voting rules

For all practical purposes, we are interested in consequential rules. But our theoretical structure allows us to determine the outcomes for all voting rules $\theta \in (0,1)$. Here, we briefly discuss the aggregation properties of the non-consequential rules.

Non-consequential rules implement the same alternative in both states under full information. We define a threshold rule θ as P-trivial (respectively, Q-trivial) if in each state, more than θ share of the population prefers P (respectively, Q).

In an ordered alternatives environment, threshold rules $\theta < F(x_L)$ are P-trivial and rules $\theta > F(x_R)$ are Q-trivial. From figure 7, it is clear that information is aggregated for all trivial rules in the ordered alternatives environment. For $\theta < F(x_L)$, $\beta_\theta^n \to 0$. Since $t(L,0) = t(R,0) = F(x_L) > \theta$, P wins in both states almost surely. Similarly, for any Q-trivial rule, $\beta_\theta^n \to 1$ and Q wins in both states almost surely. Thus, FIE holds for all $\theta \in (0,1)$ if we have ordered alternatives.

In an unordered alternatives environment, $\theta < F(x_L)$ define the P-trivial rules and $\theta > 1 - F(x_R)$ define the Q-trivial rules. For any such rule, there are three equilibrium sequences with induced priors converging to 0, β^* and 1 respectively. Information aggregation is guaranteed for any Q-trivial rule simply because $t(\omega,\beta) \leq 1 - F(x_R) < \theta$ for every β. However, this result is not true for P-trivial rules. To see this, define $t(L,\beta^*) = t(R,\beta^*) = t$.[22] For P-trivial rules $\theta > t$, *there is no equilibrium that aggregates information.* In one equilibrium sequence, $\beta_\theta^n \to \beta^*$ and the policy loses in both states. There can be additional equilibria with beliefs in $(0,\beta^*) \cup (\beta^*,1)$. In these equilibria, Q wins almost surely in one state. Figure 9 demonstrates the equilibria for one such rule.

[21] One might wonder whether FIE can fail only due to the failure of alignment. Under our assumptions, the misaligned group is never influential. Under more general preferences (for example if A1 is relaxed), it is indeed possible that the two vote share functions cross multiple times and $\theta = \theta^*(\beta)$ at some value of β where $t(L,\beta) > \theta > t(R,\beta)$. In such a case, we would get the "wrong" outcome in both states under a consequential rule.

[22] The Proof of Proposition 5 shows that $t < F(x_L)$

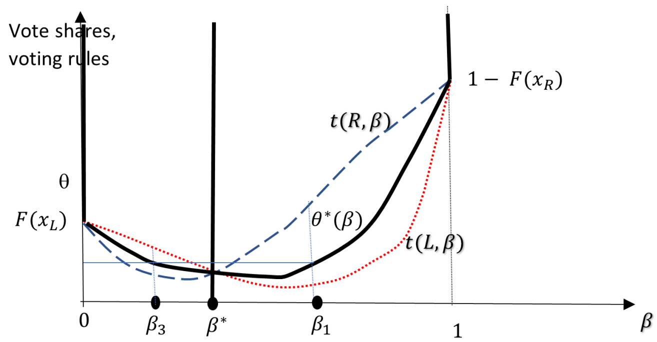


Figure 9

However, FIE holds for P-trivial rules which are low enough. For $\theta < \min_\beta \theta^*(\beta)$, equilibrium sequences here have a unique limit belief β^*, and $t(L, \beta^*) = t(R, \beta^*) = t > \theta$.

There are two important takeaways from the above discussion. First, for an unordered alternatives environment, the only voting rules that are aggregate information in all equilibrium sequences are the very high or very low thresholds which implement the same outcome in both states under full information. Moreover, there is a continuum of voting rules for which information-aggregating equilibria fail to exist. This is in contrast with the ordered alternatives environment where all non-unanimous voting rules aggregate information in every equilibrium. Second, the bias for Q shows up in a different form when we consider non-consequential rules: while FIE is guaranteed for all equilibria for Q-trivial rules, there exist P-trivial rules for which FIE fails in some or all equilibria. This again underscores how preference conflict among voters undermines aggregation of information.

6.2 Sincere Voting

We have assumed that voters are aware of the voting environment and how their actions affect the outcome. That the voters condition on being decisive follows from this presumption of voter rationality. However, pivotal voting has been criticised as a description of behavior in a stream of work (Esponda and Vespa, 2014; Margolis 2001). Now, we show that our results would not change much if we had simply assumed sincere behavior, i.e., if voters responded optimally to their exogenous prior.

Given any exogenous prior β and a consequential rule θ, information is aggregated under sincere voting if and only if $t(L, \beta) < \theta < t(R, \beta)$. Under ordered alternatives, this property holds for every $\beta \in (0, 1)$. Under unordered alternatives, FIE holds only for a for only a specific interval of priors $\beta \in (b_1, b_0)$ where b_1 solves $t(R, \beta) = \theta$ and b_0 solves $t(L, \beta) = \theta$. Since (b_1, b_0) is a subset of $(\beta^*, 1)$, $t(L, \beta) < \theta < t(R, \beta)$ for any prior in this interval. Thus, with sincere voting FIE holds for all exogenous priors under ordered alternatives but only for a specific range of priors for unordered alternatives.

6.3 Robustness

One stark feature of the model is that the outcomes are remarkably robust to variation in the parameters. For example, in the block voting and the activist voting equilibria, Q is elected even for very small noise in the signals. Similarly, these outcomes do not depend on the relative sizes of the opposing groups. In this sense, the aggregation failure arises from the *existence of* and not from the extent of state-contingent conflict in preferences.

We have assumed that a priori, the two states are equally likely. While this marginally simplifies the calculations, all that matters for the characterization is the vote share functions $t(\cdot, \cdot)$ which depend only on F and q.[23]

In the unordered alternatives case, we provide the characterization of equilibria for high signal precisions (assumption A2). While this assumption stacks the deck against non-informative equilibria and therefore makes our results sharper, we are forced to make this assumption in order to ensure that pivot probabilities are positive in each state for all prior beliefs. In absence of such an assumption, there might be additional equilibria where voters use weakly dominated strategies.[24]

It is important at this stage to discuss the role of assumption A1, which is understood to be a restriction on "curvature" of the utility function. It does not matter at all for the FIE property in the ordered alternatives environment. It is also not necessary to establish the existence of one non-aggregating outcome in the unordered alternatives environment. It is needed for providing enough regularity to the shape of the vote share function so that we can characterize the full set of equilibria. In absence of the assumption, we could have multiple crossings of the vote share functions, all of which would be equilibrium beliefs. In addition, we would then have the possibility of equilibria with wrong outcomes in each state.

We have made the assumption that $x_L^A \leq x_R^A$ for both alternatives $A \in \{P, Q\}$. This assumption restricts the number of combinations of policy-location tuples. If we had relaxed this assumption, all results would go through except for the special case where we have a "double flip". These are combinations like $x_L^P < x_R^Q < x_R^P < x_L^Q$ where in addition to unordered alternatives ($x_\omega^P - x_\omega^Q$ having different signs in different states), the P-locations and Q-locations alternate with each other. In these "double flip" cases, the monotonicity

[23] Bhattacharya (2013) provides the entire analysis with general priors.

[24] An alternative way to ensure positive pivot probabilities for all β would be to assume the existence of committed voters for each alternative. While this has been the standard assumption in the existing literature (Feddersen and Pesendorfer (1997), Bhattacharya 2013), this does not sit well with the idea of voter ideal points being distributed on the Downsian space.

condition on $-\frac{V(x,R)}{V(x,R)}$ may fail unless we make stronger assumptions on the loss functions. For instance, all our results would hold in this more general case if we had a quadratic loss function.

While there are multiple equilibria in the ordered alternative environment, we cannot think of any simple way to refine away any equilibrium sequence. In this sense, our message is that uncertainty about outcomes due to co-ordination problems is central to elections with diverse preferences. However, the non-aggregating equilibria have one focality property that the FIE equilibria do not have. In the limit of the former equilibria, the induced prior belief and strategies are independent of the particular voting rule in use. On the other hand, the FIE equilibria involve strategies and beliefs that are very sensitive to the particular value of the threshold rule θ.

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8 Proofs Omitted from the main text

8.1 Proof of Lemma 1

$EV(x|\beta_s)$ is downward sloping since $V(x,\omega)$ is decreasing in both states. Therefore, if there is some cut-off $x_s(\beta)$ satisfying $EV(x|\beta_s) = 0$, then $EV(x|\beta_s) > 0$ for $x < x_s(\beta)$ and $EV(x|\beta_s) < 0$ for $x > x_s(\beta)$. For $\beta = 1$, we have $\beta_s = 1$ and $EV(x_R|\beta_s) = V(x_R, R) = 0$ for $s \in \{l, r\}$. For $\beta = 0$, we have $\beta_s = 0$ and $EV(x_L|\beta_s) = V(x_L, L) = 0$ for $s \in \{l, r\}$. Now, consider $\beta \in (0, 1)$. The cut-off $x_s(\beta)$ is given by

$$\frac{V(x,R)}{-V(x,L)} = \frac{1-\beta_s}{\beta_s}$$

where the right hand side is a finite positive number. The left hand side takes finite positive values only in the range (x_L, x_R). Existence follows from the fact that the LHS decreases continuously from ∞ to 0 in the range of x. Moreover, as β increases, the RHS goes down and $x_s(\beta)$ must increase since the LHS is decreasing. Also, $x_r(\beta) > x_l(\beta)$ since $\beta_r > \beta_l$.

8.2 Proof of Lemma 2

The proof of Lemma 2 relies on the following property.

Property M: *Suppose a, b, c, d are numbers in $(0, 1)$ with $a < b < c < d$. There is a function $v(z)$ defined on $[0, 1]$ with $v' > 0$ and $v'' > 0$ for all $z > 0$. Assume $\frac{v''(z)}{v'(z)}$ is decreasing, i.e., v' is logconcave. Define, for $x \neq \frac{a+b}{2}$*

$$f(x) = \frac{v(|x-d|) - v(|x-c|)}{v(|x-b|) - v(|x-a|)}$$

Then $f(x)$ is increasing in $(0, \frac{a+b}{2})$.

We first verify the statement that $-\frac{V(x,R)}{V(x,L)}$ is indeed increasing in $(0, x_L) \cup (x_L, 1)$ if property M is true.

Setting $\left(x_L^P, x_L^Q, x_R^Q, x_R^P\right) = (a, b, c, d)$, we have $f(x) = -\frac{V(x,R)}{V(x,L)}$.

Property M directly tells us that $-\frac{V(x,R)}{V(x,L)}$ is increasing in $(0, x_L)$.

For $x \in (x_R, 1)$, consider $x = 1 - y$. Then, $f(x)$ equals

$$f(1-y) = g(y) = \frac{v(|1-d-y|) - v(|1-c-y|)}{v(|1-b-y|) - v(|1-a-y|)}$$

and $y \in \left(0, 1 - \frac{c+d}{2}\right)$

Setting $(a', b', c', d') = (1-d, 1-c, 1-b, 1-a)$, we have $y \in \left(0, \frac{a'+b'}{2}\right)$ and $g(y) = \frac{1}{f(y)}$. If property M is true, $g(y)$ must be decreasing in $y \in \left(0, \frac{a'+b'}{2}\right)$. Therefore, $f(x)$ must be increasing in $x \in (x_R, 1)$.

Finally, consider $f(x)$ for x in $(\frac{a+b}{2}, \frac{c+d}{2})$. Let the numerator of $f(x)$ be $n(x)$ and the denominator $d(x)$. It can be verified that (i) $n(x) > 0$ and $d(x) < 0$, and (ii) by convexity of v, $n'(x) < 0$ and $d'(x) < 0$. The sign of $f'(x)$ is the same as that of $d(x)n'(x) - n(x)d'(x)$, which is positive. This proves that $-\frac{V(x,R)}{V(x,L)}$ is increasing in the range (x_L, x_R). By continuity of $-\frac{V(x,R)}{V(x,L)}$ at $x = x_R$, Lemma 2 is true (provided property M is true).

Now we turn to the proof of property M.

We prove the result by dividing the statement of the lemma into three separate cases.

Case 1: $0 < x < a$.

$$f(x) = \frac{v(d-x) - v(c-x)}{v(b-x) - v(a-x)}$$

Now, $f'(x)$ is positive if

$$\frac{v'(d-x) - v'(c-x)}{v(d-x) - v(c-x)} < \frac{v'(b-x) - v'(a-x)}{v(b-x) - v(a-x)}$$

With some abuse of notation, denote $d - x = d,\ c - x = c,\ b - x = b$ and $a - x = a$. Notice that we still have the same ordering of a, b, c, d and they lie in $(0, 1)$.

We then need to show

$$\frac{v'(d) - v'(c)}{v(d) - v(c)} < \frac{v'(b) - v'(a)}{v(b) - v(a)} \tag{16}$$

The proof of inequality (16) relies on the following Lemma.

Lemma 5 (Lemma 2.2 in Anderson et al (1993)) *Define, for* $0 < p < q$,

$$T(q,p) = \frac{v'(q) - v'(p)}{v(q) - v(p)}$$

For any $x \in (a, b)$, $T(x, a)$ *and* $T(b, x)$ *are both decreasing in* x.

Proof.

$$\frac{dT(x,a)}{dx} = \frac{1}{[v(x) - v(a)]^2}\left[(v(x) - v(a))\, v''(x) - \big(v'(x) - v'(a)\big)\, v'(x)\right]$$

Now, by Cauchy's mean value theorem, there is some $y \in (a, x)$ such that

$$\frac{v'(x) - v'(a)}{v(x) - v(a)} = \frac{v''(y)}{v'(y)} > \frac{v''(x)}{v'(x)},$$

implying that $\frac{dT(x,a)}{dx} < 0$. Similarly,

$$\frac{dT(b,x)}{dx} = \frac{1}{[v(b) - v(x)]^2}\left[-\,(v(b) - v(x))\, v''(x) + \big(v'(b) - v'(x)\big)\, v'(x)\right]$$

Now, by Cauchy's mean value theorem, there is some $y \in (x, b)$ such that

$$\frac{v'(b) - v'(x)}{v(b) - v(x)} = \frac{v''(y)}{v'(y)} < \frac{v''(x)}{v'(x)},$$

implying that $\frac{dT(b,x)}{dx} < 0$. ■

From Lemma 5, we prove the next Lemma.

Lemma 6 *For any* $0 < p < r < q$, $T(r, p) > T(q, r)$

Proof. $T(r, p) > T(q, p)$ since $T(x, p)$ is decreasing, and $T(q, p) > T(q, r)$ since $T(q, x)$ is decreasing, by Lemma 5. ■

Now, we use the Lemma 6 to prove inequality (16). Since $0 < b < c < d$, $T(d, c) < T(c, b)$ by Lemma 6. And applying Lemma order on $0 < a < b < c$, we get $T(c, b) < T(b, a)$. Therefore, $T(d, c) < T(b, a)$. This concludes the proof of Property M for case 1.

Now we turn to the other case

Case 2: $0 < a < x < \frac{a+b}{2}$.

In this case,

$$f(x) = \frac{v(d - x) - v(c - x)}{v(b - x) - v(x - a)}$$

Now, $f'(x) > 0$ if

$$\frac{v'(d - x) - v'(c - x)}{v(d - x) - v(c - x)} < \frac{v'(b - x) + v'(a - x)}{v(b - x) - v(a - x)}$$

With some abuse of notation, denote $d - x = d$, $c - x = c$, $b - x = b$ and $x - a = a$. Notice that we still have the same ordering of a, b, c, d and they lie in $(0, 1)$.

We then need to show

$$\frac{v'(d) - v'(c)}{v(d) - v(c)} < \frac{v'(b) + v'(a)}{v(b) - v(a)} \tag{17}$$

Since $\frac{v'(b)+v'(a)}{v(b)-v(a)} > \frac{v'(b)-v'(a)}{v(b)-v(a)}$, inequality (16) implies inequality (17), which concludes the proof for case 2.

8.3 Proof of Lemma 3

The result uses the following Lemma. Define the function

$$R(s, t) = \frac{v(s) - v(t)}{s - t}$$

for $0 < s < t < 1$.

Lemma 7 *For any* p, q, r *with* $0 < p < r < q < 1$, $R(q, r) > R(q, p) > R(r, p)$

Proof. Note that $r = \alpha p + (1-\alpha) q$ for $\alpha = \frac{q-r}{q-p}$ and $0 < \alpha < 1$. By strict convexity of v, $\alpha v(p) + (1-\alpha)\, v(q) > v(r)$, or

$$\begin{aligned} v(q) - v(r) &> \alpha\{v(q) - v(p)\} = \frac{q-r}{q-p}\{v(q) - v(p)\} \\ v(r) - v(p) &< (1-\alpha)\{v(q) - v(p)\} = \frac{r-p}{q-p}\{v(q) - v(p)\} \end{aligned}$$

These two inequalities yield $R(q,r) > R(q,p)$ and $R(r,p) < R(q,p)$ respectively. ■

Next, consider (a,b,c,d) such that $0 < a < b < c < d < 1$. Then, we also have $1 > 1-a > 1-b > 1-c > 1-d > 0$.

By Lemma 7, we must have $R(d,c) > R(c,b) > R(b,a)$. Also, $R(1-a, 1-b) > R(1-b, 1-c) > R(1-c, 1-d)$. Therefore,

$$\frac{R(d,c)}{R(b,a)} > 1 > \frac{R(1-c,1-d)}{R(1-a,1-b)}$$

implying

$$\frac{v(d) - v(c)}{v(b) - v(a)} > \frac{v(1-c) - v(1-d)}{v(1-a) - v(1-b)}$$

Replacing (a,b,c,d) by $\left(x_L^P, x_L^Q, x_R^Q, x_R^P\right)$ respectively, we get our required result.

8.4 Proof of Lemma 4

Fix $s \in \{l,r\}$. For $\beta = 1$, $\beta_s = 1$ and $EV(x|\beta_s) = V(x,R) \geq 0$ iff $x \geq x_R$. Similarly, for $\beta = 0$, $EV(x|\beta_s) = V(x,L) \geq 0$ iff $x \leq x_L$. Now, consider $\beta \in (0,1)$. We have $EV(x|\beta_s) \geq 0$ iff condition (8) or condition (9) holds.

Denote $v_0 \equiv -\frac{V(0,R)}{V(0,L)}$ and $v_1 \equiv -\frac{V(1,R)}{V(1,L)}$. By Lemma 2, $-\frac{V(x,R)}{V(x,L)} \in [v_0, \infty)$ when $x \in [0, x_L]$ and $-\frac{V(x,R)}{V(x,L)} \in [0, v_1]$ when $x \in [x_R, 1]$. Lemma 3 states that $v_0 > v_1$, implying that $x \to -\frac{V(x,R)}{V(x,L)}$ is one-to-one. Hence, for β such that $\frac{1-\beta_s}{\beta_s} \in [0, v_1] \cup [v_0, \infty)$, there is a unique solution to $-\frac{V(x,R)}{V(x,L)} = \frac{1-\beta_s}{\beta_s}$ given by $x_s(\beta)$.

Consider now some β such that $\frac{1-\beta_s}{\beta_s} = v \in [v_0, \infty)$. The corresponding $x_s(\beta)$ lies in $[0, x_L]$. By (8), $x < x_s(\beta)$ votes for P and $x \in (x_s(\beta), x_L]$ votes for Q due to monotonicity of $\frac{V(x,R)}{V(x,L)}$ in $[0, x_L]$. Since $\frac{V(x,R)}{V(x,L)} < v_1 < v$ for all $x \geq x_R$, all types $x \geq x_R$ vote for Q by (9). All types (x_L, x_R) vote for Q since both $EV(x|\beta_s) < 0$ in that range for all β_s. This establishes that for $\frac{1-\beta_s}{\beta_s} = v \in [v_0, \infty)$, there is a type $x_s(\beta)$ such that $x \leq x_s(\beta)$ votes for P and $x > x_s(\beta)$ votes for Q. Since $-\frac{V(x,R)}{V(x,L)}$ is increasing and $\frac{1-\beta_s}{\beta_s}$ is decreasing in β, $x_s(\beta)$ must be a decreasing function as long as $\frac{1-\beta_s}{\beta_s} \in [v_0, \infty)$. Let $\underline{\beta}^s$ be the value of β for which $\frac{1-\underline{\beta}^s}{\underline{\beta}^s} = v_0$. Then, by continuity, there is an interval $[0, \underline{\beta}^s]$ of β for which $\frac{1-\beta_s}{\beta_s} \in [v_0, \infty)$. This completes part (a) of the Lemma.

For part (b), define $\overline{\beta^s}$ as that value of β for which $\frac{1-\beta^s}{\beta^s} = v_1$. Since $v_0 > v_1$, it must be that $\overline{\beta^s} > \underline{\beta^s}$. For $\beta \in (\underline{\beta^s}, \overline{\beta^s})$, $\frac{1-\beta_s}{\beta_s} \in (v_1, v_0)$. Consider $\frac{1-\beta_s}{\beta_s} = v' \in (v_1, v_0)$. For all $x \geq x_R$, $-\frac{V(x,R)}{V(x,L)} < v' = \frac{1-\beta_s}{\beta_s}$, so no type $x \geq x_R$ votes for P by (9). Similarly, by (8), no type $x \leq x_L$ votes for P. All types (x_L, x_R) vote for Q for every belief.

Part (c) follows in the same way as part (a).

Finally, notice that for any $\beta \in (0,1)$, $\beta^r > \beta^l$, or $\frac{1-\beta_r}{\beta_r} < \frac{1-\beta_l}{\beta_l}$. This implies that $\overline{\beta^r} < \overline{\beta^l}$, $\underline{\beta^r} < \underline{\beta^l}$ and $x_r(\beta) < x_l(\beta)$ whenever both cut-offs are in the same group, i.e., $\beta \in (0, \underline{\beta^r}) \cup (\overline{\beta^l}, 1)$.

8.5 Proof of Claim 1

We will now express $\overline{\beta^r}$ and $\underline{\beta^l}$ as functions of q. Consider the signal precision q for which $\overline{\beta^r}(q) = \underline{\beta^l}(q) = \widehat{\beta}$ Then,

$$\frac{1-\beta_r}{\beta_r} = \frac{(1-q)(1-\widehat{\beta})}{q\widehat{\beta}} = v_1$$

$$\frac{1-\beta_l}{\beta_l} = \frac{q(1-\widehat{\beta})}{(1-q)\widehat{\beta}} = v_0$$

where $v_0 \equiv -\frac{V(0,R)}{V(0,L)}$ and $v_1 \equiv -\frac{V(1,R)}{V(1,L)}$. By eliminating $\widehat{\beta}$, we get

$$\frac{q}{1-q}v_1 = \frac{1-q}{q}v_0 \text{ or } q = \frac{1}{1+\sqrt{\frac{v_1}{v_0}}} \equiv \widehat{q} \in \left(\frac{1}{2}, 1\right),$$

since $v_1 < v_0$ by Lemma 3.

For $q > \widehat{q}$, given $\beta = \widehat{\beta}$, we must have $\frac{1-\beta_r}{\beta_r} = \frac{(1-q)(1-\widehat{\beta})}{q\widehat{\beta}} < \frac{(1-\widehat{q})(1-\widehat{\beta})}{\widehat{q}\widehat{\beta}} = v_1$. Hence, $\overline{\beta^r}(q) < \widehat{\beta}$. Similarly, $\underline{\beta^l}(q) > \widehat{\beta}$, implying $\overline{\beta^r}(q) < \underline{\beta^l}(q)$.

8.6 Proof of Proposition 2

The vote share functions $t(\omega, \beta)$ for $\omega \in \{0,1\}$ are given by (5) and (6), and the functions $z_s(\beta)$, $s \in \{l, r\}$ by (10). By claim (1) we have $0 < \underline{\beta^r} < \overline{\beta^r} < \underline{\beta^l} < \overline{\beta^l} < 1$. In $\beta \in [0, \underline{\beta^r})$, $z_s(\beta)$ is decreasing for both $s \in \{l, r\}$ by Lemma 4. In the range $\beta \in [\underline{\beta^r}, \overline{\beta^r}]$, $z_l(\beta)$ is decreasing while $z_r(\beta) = 0$. Therefore, for $\beta \in [0, \overline{\beta^r}]$, both $t(L, \beta)$ and $t(R, \beta)$ are decreasing.

Also, $t(L,\beta) - t(R,\beta) = (2q-1)(z_l(\beta) - z_r(\beta))$. In $\beta \in [0, \underline{\beta^r})$, $z_l(\beta) - z_r(\beta) > 0$ since $0 \leq x_l(\beta) < x_r(\beta)$. In $\beta \in [\underline{\beta^r}, \overline{\beta^r}]$, $z_l(\beta) > 0$ and $z_r(\beta) = 0$. Hence, for $\beta \in [0, \overline{\beta^r}]$, $t(L, \beta) > t(R, \beta)$.

For analogous reasons, in the range $[\overline{\beta^l}, 1]$, $t(L,\beta)$ and $t(R,\beta)$ are both upward sloping and $t(L,\beta) < t(R,\beta)$.

Finally, in the range $(\overline{\beta^r}, \underline{\beta^l})$, $z_l(\beta)$ is strictly decreasing and goes from $F(x_l(\overline{\beta^r})) > 0$ to 0, while $z_r(\beta)$ is increasing and goes from 0 to $1 - F(x_r(\underline{\beta^l})) > 0$. $t(L,\beta) - t(R,\beta) = (2q-1)(z_l(\beta) - z_r(\beta))$ is therefore a decreasing function which must also take value zero at some unique $\beta^* \in (\overline{\beta^r}, \underline{\beta^l})$.

8.7 Proof of Remark 1

Continuity of $\theta^*(\beta)$ follows from continuity of vote share functions. Now, denote $t(R,\beta) = x$ and $t(L,\beta) = y$, and assume WLOG $0 < y < x < 1$. For any $\theta \in (0,1)$, the function $f(z) = z^\theta(1-z)^{1-\theta}$ is single-peaked in $[0,1]$ and achieves maximum at $z = \theta$. Therefore, the equation $x^\theta(1-x)^{1-\theta} = y^\theta(1-y)^{1-\theta}$ has a unique solution θ^* and $y < \theta^* < x$.

Consider now the expression for θ^*.

$$\frac{\theta^*}{1-\theta^*} = \frac{\log(1-y) - \log(1-x)}{\log x - \log y}$$

Denote the right hand side by $h(x,y)$. We have

$$\frac{\partial h}{\partial x} = \frac{1}{(\log x - \log y)^2}\left[\frac{1}{1-x}(\log x - \log y) - \frac{1}{x}\log(1-y) - \log(1-x)\right]$$

Thus $\frac{\partial h}{\partial x} > 0$ if $\frac{x}{1-x} > \frac{\theta^*}{1-\theta^*}$, i.e., $x > \theta^*$. Similarly, $\frac{\partial h}{\partial y} > 0$ if $y < \theta^*$. Since we have $y < \theta^* < x$, $h(x,y)$ is increasing in both x and y. Since $\frac{\theta^*}{1-\theta^*}$ is itself increasing in θ^*, it must be increasing in both x and y.

8.8 Proof of Proposition 4

Consider $\theta \in (F(x_L), F(x_R))$. By Proposition 3, the only way θ can belong to $\Theta^O(\beta)$ would be if $\beta \in (0,1)$. By Proposition 1 and remark 1, $\theta^*(\beta)$ is continuous and monotonically increasing in the range $(0,1)$. Moreover, $\theta^*(\beta) \to F(x_L) < \theta$ as $\beta \to 0$ and $\theta^*(\beta) \to F(x_R) > \theta$ as $\beta \to 1$. Thus, there is a uniuqe value β_a such that $\theta^*(\beta_a) = \theta$. Since $\theta^*(\cdot)$ is increasing, β_a is regular. Therefore, for all equilibrium sequences, $\beta_\theta^n \to \beta_a$. Notice that $t(L,\beta_a) < \theta < t(R,\beta_a)$ by Proposition 1. By the Strong Law of Large numbers, P wins in state R and Q in state L almost surely.

8.9 Proof of Proposition 5

Consider a voting rule $\theta \in (F(x_L), 1 - F(x_R))$. We identify the three classes of equilibrium sequences in (i), (ii) and (iii) we establish the existence of equilibrium sequences converging to 1, β^* and some $\beta \in (\beta^*, 1)$ respectively. In (iv) we establish that there is no equilibrium sequence that converges to any belief in $(0,\beta^*) \cup \{1\}$.

(i) From Proposition 3, $\theta \in \Theta^U(0)$. Since $t(L,0) = t(R,0) = F(x_L) > \theta$, $\beta = 0$ is regular. Therefore there is an equilibrium sequence with $\beta_n^\theta \to 0$, and in the limit of this sequence, Q wins almost surely in both states.

(ii) From Proposition 3, we also have $\theta \in \Theta^U(\beta^*)$. Observe that $t(L,\beta^*) = t(R,\beta^*) < F(x_L)$, since $t(\omega,\beta^*) = z_l(\beta^*)$ and $F(x_L) = z_l(0)$, and z_l increases in the range $(0, \overline{\beta^l})$. Since $\theta > F(x_L)$, β^* is regular and there is an equilibrium sequence $\beta_n^\theta \to \beta^*$ and in the limit of this sequence, Q wins almost surely in both states.

(iii) To show that there some $\beta \in (\beta^*, 1)$ such that there is an equilibrium sequence that converges to β, we proceed in two steps. First, we show that there is some $\beta \in (\beta^*, 1)$ such that the necessary condition $\theta = \Theta^U(\beta)$ is met. Then we show that among the possibly multiple values of $\beta \in (\beta^*, 1)$ satisfying the necessary condition, there must exist some β that satisfies the sufficiency condition too.

For necessity, observe that (a) $\theta^*(\beta)$ is continuous in the range $(\beta^*, 1)$, (b) $\theta^*(\beta) \to 1 - F(x_R) > \theta$ as $\beta \to 1$, and (c) $\theta^*(\beta) \to t < F(x_L) < \theta$ as $\beta \to \beta^*$. Therefore, there must be some β_1 in the range $(0, \beta^*)$ such that $\theta^*(\beta_1) = \theta$.

To check for sufficiency, note that conditions (a), (b) and (c) above imply that *some* solution to $\theta^*(\beta_1) = \theta$ must satisfy one of the following two properties: (I) $\theta^*(\cdot)$ is locally increasing at β_1, or (II) there is some interval $[\beta_1', \beta_1''] \subset (\beta^*, \underline{\beta^l})$ containing β_1 such that $\theta^*(\cdot) = \theta$ in $[\beta_1', \beta_1'']$, and for some $\varepsilon > 0$, $\theta^*(\cdot)$ is increasing in $(\beta_1' - \varepsilon, \beta_1')$ as well as in $(\beta_1'', \beta_1'' + \varepsilon)$.

If property (I) holds, then β_1 is regular and we are done. It is, however, possible that the only values of β for which $\theta^*(\cdot) = \theta$ are not regular because $\theta^*(\cdot)$ is constant over some interval $[\beta_1', \beta_1'']$. In this case, property (II) must hold. We now show that if property (II) holds, then there must exist an equilibrium sequence with β_n^θ converging to some value in $[\beta_1', \beta_1'']$.

Denote $t(R,\beta) = x$, $t(L,\beta) = y$, $g(x,y,\theta) = \frac{x^\theta(1-x)^{1-\theta}}{y^\theta(1-y)^{1-\theta}}$ and $B(\beta) = \left[\frac{1-x}{1-y}\right]^{n-m-1}$ where $m = \frac{\lfloor n\theta \rfloor}{\theta}$. We can now write $H(\beta,n,\theta)$ as $B(\beta)\,[g(x,y,\theta)]^m$. Notice that in the interval $[\beta_1' - \varepsilon, \beta_1'' + \varepsilon]$, $0 < y < x < 1$. In $[\beta_1', \beta_1'']$, $g(x,y,\theta) = 1$ since $\theta = \theta^*$ for this interval. Therefore, $H(\beta,n,\theta) = B(\beta)$. We have $G_n(\beta,\theta) = \frac{B(\beta)}{1+B(\beta)}$, which is bounded above 0 and below 1 for each n. By an argument employed before in the the Sufficiency Lemma in the proof of Proposition 3, we have $G_n(\beta,\theta) \to 1$ for all $\beta \in [\beta_1' - \varepsilon', \beta_1')$, and $G_n(\beta,\theta) \to 0$ for all $\beta \in (\beta_1'', \beta_1'' + \varepsilon']$ for $\varepsilon' > 0$ small enough. Notice also that $G_n(\beta,\theta)$ is continuous in β for all θ in $[\beta_1', \beta_1'']$. For large enough n, $G_n(\beta,\theta) - \beta > 0$ at $\beta_1' - \varepsilon'$ and $G_n(\beta,\theta) - \beta < 0$ at $\beta_1'' + \varepsilon'$. Therefore, there must exist a fixed point $\beta_n \in (\beta_1', \beta_1'')$ for all n large enough. Since (β_1', β_1'') is contained in a closed and bounded interval $[\beta_1', \beta_1'']$, a limit point must also exist and lie in $[\beta_1', \beta_1'']$, and we are done.

We thus establish the existence of an equilibrium sequence with induced beliefs converging to $\beta_1 \in (\beta^*, 1)$. By Proposition 2, $t(L,\beta_1) < \theta < t(R,\beta_1)$ in $(0,\beta^*)$ and P wins in state R and Q in state L almost surely.

It remains to show that we cannot have any equlibrium sequence with induced prior converging to values in $(0, \beta^*) \cup \{1\}$. For all β in $(0, \beta^*)$, $\theta^*(\beta) < t(L, \beta) < z_l(\beta) < F(x_L)$, where the last inequality follows from the fact that $z_l(\beta)$ is decreasing in that range and $z_l(0) = F(x_L)$. Since $\theta > F(x_L)$, there is no β in $(0, \beta^*)$ for which $\theta = \theta^*(\beta)$. Finally, observe that $\theta \notin \Theta^U(1)$, and we are done.

8.10 Proof of Remark 2

At $\beta = \underline{\beta}^l$, $\theta^*(\beta) < t(R, \beta) = qz_r(\underline{\beta}^l) = q\left(1 - F\left(x_r\left(\underline{\beta}^l\right)\right)\right)$. By the condition that $F(x_L) \geq q\left(1 - F(x_r\left(\underline{\beta}^l\right)\right)$, it must be the case that $\theta > \theta^*(\underline{\beta}^l)$, since $\theta > F(x_L)$. On the other hand, we have $\theta < \theta^*(\beta)$ for $\beta \to 1$ since $\theta < 1 - F(x_R)$. By Remark 1, $\theta^*(\beta)$ is strictly increasing in $(\underline{\beta}^l, 1)$. Therefore, there is a unique solution β_1 to $\theta = \theta^*(\beta)$ in the range $(\underline{\beta}^l, 1)$. Moreover, there is no solution to $\theta^*(\beta) = \theta$ in $(\beta^*, \underline{\beta}^l]$. At such a β_1, property (I) holds, and the rest follows from the proof of Proposition 5.

9 Supplementary Online Appendix

In this appendix, we provide the proof of Proposition Proof of Proposition 3. We prove this proposition in several steps. We start with Lemma 1 in Bhattacharya (2013).

Lemma 8 *For any given environment, if the limit belief in an equilibrium sequence is β_θ^0, then*

$$\left(t(L, \beta_\theta^0)\right)^\theta \left(1 - t(L, \beta_\theta^0)\right)^{1-\theta} = \left(t(R, \beta_\theta^0)\right)^\theta \left(1 - t(R, \beta_\theta^0)\right)^{1-\theta}$$

Proof. Note first, that by the usual continuity arguments, if $\beta_\theta^n \to \beta_\theta^0$, then $t(\omega, \beta_\theta^n) \to t(\omega, \beta_\theta^0)$. Whenever $t(\omega, \beta_\theta^0) = t(\omega, \beta_\theta^0)$, the Lemma holds trivially. Notice that this covers the cases $\beta_\theta^0 \in \{0, 1\}$ ordered alternatives and $\beta_\theta^0 \in \{0, \beta^*, 1\}$ for unordered alternatives. Now, consider $\beta_\theta^0 \in (0, 1)$ and rewrite $H(\beta_\theta^n, n, \theta)$ as $\left[\frac{1-t(R,\beta_\theta^n)}{1-t(L,\beta_\theta^n)}\right]^{n-m-1} \left[\frac{t(R,\beta_\theta^n)^\theta\left(1-t(R,\beta_\theta^n)\right)^{1-\theta}}{t(L,\beta_\theta^n)^\theta\left(1-t(L,\beta_\theta^n)\right)^{1-\theta}}\right]^m$ where $m = \frac{\lfloor n\theta \rfloor}{\theta}$. Since $m \geq n - \frac{1}{\theta}$, we have $m \to \infty$ as $n \to \infty$. Also, since $0 < t(\omega, \beta) < 1$ and $m - n \in \left[0, \frac{1}{\theta}\right]$, there is some $0 < \underline{t} < \bar{t}$ such that $\underline{t} \leq \left[\frac{1-t(R,\beta_\theta^n)}{1-t(L,\beta_\theta^n)}\right]^{n-m-1} \leq \bar{t}$ for all m and n. If there is some $\varepsilon > 0$ such that $\frac{t(R,\beta_\theta^n)^\theta\left(1-t(R,\beta_\theta^n)\right)^{1-\theta}}{t(L,\beta_\theta^n)^\theta\left(1-t(L,\beta_\theta^n)\right)^{1-\theta}} > 1 + \varepsilon$ for all n large enough, then $\lim_{n\to\infty} H(\beta_\theta^n, n, \theta) > \lim_{n\to\infty} \underline{t} \left[\frac{t(R,\beta_\theta^n)^\theta\left(1-t(R,\beta_\theta^n)\right)^{1-\theta}}{t(L,\beta_\theta^n)^\theta\left(1-t(L,\beta_\theta^n)\right)^{1-\theta}}\right]^{\frac{\lfloor n\theta \rfloor}{\theta}} > \underline{t}\left[\lim_{m\to\infty} (1 + \varepsilon)^m\right] \to \infty$. Hence the RHS of equation (13) is not bounded away from 1, which is a contradiction. Similarly, if there is some $\varepsilon > 0$ such that $\frac{t(R,\beta_\theta^n)^\theta\left(1-t(R,\beta_\theta^n)\right)^{1-\theta}}{t(L,\beta_\theta^n)^\theta\left(1-t(L,\beta_\theta^n)\right)^{1-\theta}} < 1 - \varepsilon$ for all n large enough, then $\lim_{n\to\infty} H(\beta_\theta^n, n, \theta) = 0$, we have a contraction since the RHS of equation (13) is not bounded away from 0. ■

Next, we state a version of Lemma 2 in Bhattacharya (2013) adapted to our setting.

Lemma 9 (NECESSITY) *Define by* $\Theta(\beta)$ *for each environment as given in Proposition 3. For a given* $\beta \in [0,1]$, *consider any sequence* $\beta^n \to \beta$. *Now, if* $\theta \notin \Theta(\beta)$, *then* $H(\beta^n, n, \theta)$ *is bounded away from* $\frac{\beta}{1-\beta}$ *as* $n \to \infty$.

Proof. Consider the environment with ordered alternatives first.

The necessity of case (i) follows directly from Lemma 8. Next, consider case (ii), i.e., $\Theta(0) = \{\theta : \theta \leq F(x_L)\}$. From Proposition 1, by continuity of vote share functions, we know that as $\beta^n \to 0$, both $t(L,\beta^n)$ and $t(R,\beta^n)$ converge to $F(x_L)$ and $F(x_L) < t(L,\beta^n) < t(R,\beta^n)$. Notice now that for $z \in (0,1)$, the the function $z^\theta(1-z)^{1-\theta}$ is single peaked in z and attains its maximum at $z = \theta$. Now, if $\theta > F(x_L)$, then for all large enough n, we have $\theta > t(R,\beta^n) > t(L,\beta^n)$, which would imply that $(t(R,\beta^n))^\theta (1-t(R,\beta^n))^{1-\theta} > (t(L,\beta^n))^\theta (1-t(L,\beta^n))^{1-\theta}$. Therefore, we must have $H(\beta^n, n, \theta) > \underline{t} \left[\frac{t(R,\beta^n)^\theta(1-t(R,\beta^n))^{1-\theta}}{t(L,\beta^n)^\theta(1-t(L,\beta^n))^{1-\theta}}\right]^{\frac{\lfloor n\theta \rfloor}{\theta}} > \underline{t}$, which is bounded away from 0 in the limit. The proof of case (iii) is analogous.

Next, consider the environment with unordered alternatives. Case (i) follows from Lemma 8. In case (ii), there is nothing to prove. Cases (iii) and (iv) are analogous to case (ii) of the ordered alternatives case. ■

Our sufficiency result follows Lemma 3 in Bhattacharya (2013). We need a slightly more relaxed concept of regularity in the present paper.

Lemma 10 (SUFFICIENCY) *There is a sequence of equilibria with induced prior* β_θ^n *converging to* β *if* $\theta \in \Theta(\beta)$ *and* β *is regular.*

Proof. Define the function $G_n(\beta,\theta) = \frac{H(\beta,n,\theta)}{1+H(\beta,n,\theta)}$. We show that provided that $\widehat{\beta}$ is regular, if $\theta \in \Theta(\widehat{\beta})$, then there is a sequence of fixed points β_θ^n of $G_n(\beta,\theta)$ such that $\beta_n^\theta \to \widehat{\beta}$. We prove this separately for different values of $\widehat{\beta}$.

We use the following result repeatedly in the proof: Suppose $g(x,y,\theta) = \frac{x^\theta(1-x)^{1-\theta}}{y^\theta(1-y)^{1-\theta}}$ for some $1 > x > y > 0$. Then, we must have $\frac{\partial g(x,y,\theta)}{\partial \theta} > 0$.

First consider some regular $\widehat{\beta}$ such that $t(L,\widehat{\beta}) \neq t(R,\widehat{\beta})$. WLOG, assume $t(R,\widehat{\beta}) > t(L,\widehat{\beta})$. For such a $\widehat{\beta}$, $\Theta(\widehat{\beta}) = \theta^*(\widehat{\beta})$ by Lemma 9 and notice that $\theta^*(\cdot)$ has continuous and bounded derivatives since f is continuous and bounded. Since $\widehat{\beta}$ is regular, there must be a neighborhood $(\widehat{\beta}-\varepsilon, \widehat{\beta}+\varepsilon)$ where $\theta^*(\beta)$ is either only increasing or only decreasing, and because f is bounded, $t(R,\beta) > t(L,\beta)$. Suppose first that $\theta^*(\cdot)$ is decreasing in $(\widehat{\beta}-\varepsilon, \widehat{\beta}+\varepsilon)$. Write $H(\beta,n,\theta)$ as $B(\beta)\,[g(x,y,\theta)]^m$ where $x = t(R,\beta)$, $y = t(L,\beta)$, $B(\beta) = \left[\frac{1-x}{1-y}\right]^{n-m-1}$ and $m = \frac{\lfloor n\theta \rfloor}{\theta}$. $B(\beta)$ is bounded above and below. Now, for $\beta \in (\widehat{\beta}, \widehat{\beta}+\varepsilon)$, we must have $g(x,y,\theta^*(\widehat{\beta})) > 1$, since $\theta^*(\widehat{\beta}) > \theta^*(\beta)$ as $\theta^*(\cdot)$ is decreasing. Moreover, $g(x,y,\theta^*(\beta)) = 1$ by definition. As $n \to \infty$, m must also go to ∞, and then, $\left[g(x,y,\theta^*(\widehat{\beta}))\right]^m \to \infty$, implying that $H(\beta,n,\theta^*(\widehat{\beta})) \to \infty$, i.e., $G_n(\beta,\theta^*(\widehat{\beta})) \to 1$.

We have just shown that for $\beta \in (\widehat{\beta}, \widehat{\beta}+\varepsilon)$, we must have $G_n(\beta, \theta^*(\widehat{\beta})) \to 1$ as $n \to \infty$. On the other hand, for $\beta \in (\widehat{\beta}-\varepsilon, \widehat{\beta})$, we must have $G_n(\beta, \theta^*(\widehat{\beta})) \to 0$ as $n \to \infty$. Consider the (continuous) function $G_n(\beta, \theta^*(\widehat{\beta})) - \beta$ in the range $\beta \in (\widehat{\beta}-\varepsilon, \widehat{\beta}+\varepsilon)$. Given ε, for large enough n, it is positive for $\beta = \widehat{\beta}+\varepsilon$, and negative for $\beta = \widehat{\beta}-\varepsilon$. Thus, there must exist some $\beta_n^{\theta^*(\widehat{\beta})} \in (\widehat{\beta}-\varepsilon, \widehat{\beta}+\varepsilon)$ such that $G_n\left(\beta_n^{\theta^*(\widehat{\beta})}, \theta^*(\widehat{\beta})\right) - \beta_n^{\theta^*(\widehat{\beta})} = 0$ for all n large enough. Thus, there exists a sequence $\beta_n^{\theta^*(\widehat{\beta})}$ such that for any $\varepsilon > 0$ small enough, there is some M such that for all $n > M$, $G_n\left(\beta_n^{\theta^*(\widehat{\beta})}, \theta^*(\widehat{\beta})\right) = \beta_n^{\theta^*(\widehat{\beta})}$ and $\left|\beta_n^{\theta^*(\widehat{\beta})} - \widehat{\beta}\right| < \varepsilon$. If $\theta^*(\beta)$ is increasing in $(\widehat{\beta}-\varepsilon, \widehat{\beta}+\varepsilon)$, then we can prove the result in an analogous way.

Next, consider $\widehat{\beta} = \beta^* \neq \frac{1}{2}$, and denote $t(L, \beta^*) = t(R, \beta^*) = t$. WLOG, suppose first that $\beta^* > \frac{1}{2}$. Since $t(L, \widehat{\beta}) = t(R, \widehat{\beta})$, $G_n(\widehat{\beta}, \theta) = \frac{1}{2}$ for all (n, θ). Then, consider any $\theta > t$. Since by proposition 2, $t(R, \beta) > \theta^*(\beta) > t(L, \beta)$ for all $\beta \in (\widehat{\beta}, \widehat{\beta}+\varepsilon)$, given θ we can choose ε small enough such that $\theta > \theta^*(\beta)$ for all $\beta \in (\widehat{\beta}, \widehat{\beta}+\varepsilon)$. Therefore $G_n(\beta, \theta) \to 1$ in this interval. Now, consider the continuous function $G_n(\beta, \theta) - \beta$ in this interval. For large enough n, it is positive at $\widehat{\beta}+\varepsilon$ and negative at $\widehat{\beta}$. Therefore, $G_n(\beta, \theta)$ must have a fixed point β_n in this interval. Thus, there exists a sequence β_n such that for any $\varepsilon > 0$ small enough, there is some m such that for all $n > m$, $G_n(\beta_n, \theta) = \beta_n$ and $\left|\beta_n - \widehat{\beta}\right| < \varepsilon$ for any $\theta > t$. To show the existence of a sequence of beliefs converging to $\widehat{\beta}$ for voting rules $\theta < t$, follow an analogous method.

If $\widehat{\beta} = \beta^* = \frac{1}{2}$, consider a sequence $\beta_n = \widehat{\beta}$ for all n. We are done, since $G_n(\widehat{\beta}, \theta) = \frac{1}{2} = \widehat{\beta}$ for all n.

Finally, consider the cases with $\widehat{\beta} \in \{0, 1\}$. Suppose first that $\widehat{\beta} = 1$. For the ordered alternatives case, consider any $\theta > 1 - F(x_R)$. Now, by proposition 1 we must have $\theta > t(R, \beta) > \theta^*(\beta) > t(L, \beta)$ for small enough $(0, \varepsilon)$. Notice that $G_n(\beta, \theta) \to 1$ in this interval. The function $G_n(\beta, \theta) - \beta$ is equal to $-\frac{1}{2}$ at $\beta = 1$ and positive (close to ε) at $\beta = \varepsilon$ for large enough n. Therefore, there must be a fixed point of $G_n(\beta, \theta)$ in $(0, \varepsilon)$. The other cases with $\widehat{\beta} \in \{0, 1\}$ are similar. ■